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NAVAL RESEARCH LOGISTICS QUARTERLY

SEPTEMBER 1969
VOL. 16, NO. 3

OFFICE OF NAVAL RESEARCH

NAVSO P-1278

407-B

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The Naval Research Logistics Quarterly is published by the Office of Naval Research in the months of March June, September, and December and can be purchased from the Superintendent of Documents, U.S. Govern ment Printing Office, Washington, D.C. 20402. Subscription Price: $5.50 a year in the U.S. and Canada, $7.0 elsewhere. Cost of individual issues may be obtained from the Superintendent of Documents.

The views and opinions expressed in this quarterly are those of the authors and not necessarily those of the Offic of Naval Research.

Issuance of this periodical approved in accordance with Department of the Navy Publications and Printing Regulation NAVEXOS P–35

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DETERMINING OPTIMUM REJECT ALLOWANCES FOR DETERIORATING PRODUCTION SYSTEMS*†

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ABSTRACT

Since most manufacturing processes inevitably produce some defective items, it is common practice to produce a quantity larger than the actual order size. This excess is called a reject allowance. In this paper we consider production processes which undergo stochastic deterioration and demonstrate that under appropriate conditions a smallest optimum reject allowance exists and can be easily computed. We also investigate the optimality of periodic process inspections and show that in some cases one can do better by following an (r, R) inspection-processing policy.

1. INTRODUCTION AND GENERAL DESCRIPTION OF THE PROBLEM

Suppose that a production facility is asked to have available a fixed quantity of non-defective items. Since most manufacturing processes inevitably produce some defective items, it is common practice to produce a quantity larger than the actual order size. This excess is called a reject allowance and it provides for producing enough extra items to replace the defectives. The selection of the most appropriate reject allowance requires an explicit evaluation of the relative seriousness of overage and underage as well as an analysis of the probability mass function for the number of defectives generated. Also, this allowance has to be chosen before the production starts.

The cost of processing a batch through the production facility consists of a fixed setup cost and a direct cost proportional to the quantity processed. It is assumed that the order size is prescribed by a single number without any margin for error. Therefore, producing a sufficient number of non-defective items to fill the order size is mandatory and any shortage must be made up through one or more supplementary runs reincurring the fixed setup cost every time. Furthermore, any extra non-defectives have little or no salvage value.

Several methods have been proposed for computing reject allowances for non-deteriorating production systems. Earlier efforts by Bowman and Fetter [1957], Llewellyn [7], and Giffler [1] use a very simplified cost structure and ignore all underage costs, except the fixed cost associated with one extra setup. The first two also assume process quality to be independent of the quantity produced and use binomial or Poisson density functions in their model. Giffler, however, assumed process quality to be a function of the quantity produced and gave a sequential algorithm to solve the problem. Levitan [6] also considered a similar model and derived conditions for a unique optimum to exist, thereby

*The results presented in this paper are part of the author's doctoral dissertation submitted to the Department of Industrial Engineering, Stanford University. The author is grateful to Prof. Frederick S. Hillier for his guidance during the course of this investigation.

†This work was supported in part by the Office of Naval Research under Contract Nonr–225(89). Reproduction in whole or in part is permitted for any purpose of the United States Government.

providing a mathematical justification for Giffler's results and algorithm. The method proposed by Goode and Saltzman [2] uses a comprehensive cost model to obtain a reasonably exact solution. Unfortunately, the computations involved are extremely extensive except for very small order quantities. Hillier [4] used the same cost model but by using certain standard mathematical techniques—primarily, taking the first differences and thereby using incremental analysis rather than total value analysis—obtained a simplified computational procedure which compares favorably with the procedures for approximate methods. Wadsworth and Chang [9] used similar analysis, but assumed process quality to be a function of the quantity produced. By making some simplifying assumptions, they derived conditions for optimality and gave a simple algorithm to calculate optimum reject allowance. Recently, Gregory and Dov [3] gave a formulation using a rather specialized probability distribution for the number of defectives. Although their model is rather restrictive, it may have some interesting and useful applications and extensions.

In this paper, our main point of departure is to assume that the production process undergoes stochastic deterioration. This case is of great practical importance since most production systems deteriorate over time and need periodic inspections, maintenance, and replacements. Moreover, this deterioration over time is predictable only in terms of probabilities. In order to render the analysis mathematically tractable, we assume the following simple model for the deteriorating system. Suppose that the system is in state 1, called the "*good state*," immediately after the process setup. As each item is being processed there is a fixed probability, say $\alpha(0<\alpha<1)$, of the process transferring to state 2, called the "*bad state*." Once the process has deteriorated to state 2, it stays there unless repaired and transferred back to state 1. In either state, processing each item is a Bernoulli trial with a fixed probability of the item being defective. Let these probabilities be p_1 and p_2 for states 1 and 2 respectively, where $p_1<p_2$. Furthermore, it is possible to determine the state of the production system through periodic inspections.

For the purposes of this paper, we assume that $p_1=0$ and $p_2=p(0<p\leq 1)$. Thus, when the production process is in state 1, it produces *no* defective items and when the process transfers to state 2, the number of defectives generated by it obeys a binomial probability law with parameter p. The probability mass function for the number of defectives is

$$\phi(x,N)=\begin{cases}\alpha[q^N+\bar{\alpha}q^{N-1}+\ \cdots\ +\bar{\alpha}^{N-1}q]+\bar{\alpha}^N, & \text{for } x=0\\ \alpha p^x\left[\binom{N}{x}q^{N-x}+\binom{N-1}{x}\bar{\alpha}q^{N-x-1}+\ \cdots\ +\binom{x+1}{x}\bar{\alpha}^{N-x-1}q+\bar{\alpha}^{N-x}\right], & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x=1,2,\ \ldots\ ,N-1 & \\ \alpha p^N, \text{ for } x=N, & \end{cases} \tag{1.1}$$

where

$\phi(x,N)=$ probability of obtaining x defectives when N items are processed,

$\bar{\alpha}=1-\alpha$,

$q=1-p$.

Furthermore, the expected value of the number of defectives is

$$\begin{aligned}\bar{x}(N)&=\sum_{x=1}^{N}x\phi(x,N)\\ &=Np-p\frac{\bar{\alpha}}{\alpha}(1-\bar{\alpha}^N).\end{aligned} \tag{1.2}$$

We begin by assuming that intermediate process inspections are not allowed and compute the optimum starting quantity which minimizes the expected total cost. Next, we allow periodic inspections of the process and determine the optimal inspection interval, called the "economic lot size." We then compute the number of such lots to be processed in order to satisfy the customer's order.

2. OPTIMUM SIZE OF THE STARTING QUANTITY: NO INTERMEDIATE PROCESS INSPECTIONS ALLOWED.

We shall develop an expression for the expected total cost and then perform incremental analysis by taking first differences to determine the optimum starting quantity. Let:

C_s = setup cost,
C_u = direct cost per item,
L = number of items required to fill the order, i.e., the order size,
N = number of items started in first production run,
$A = N - L$, i.e., allowance or number of excess items started in first run,
$TC_{N,L}$ = expected total cost given N and L, and
TC_L^* = minimum value of $TC_{N,L}$ over all values of N.

Since there is no process inspection during the course of the production run, the expected total cost is easily seen to be

$$TC_{N,L} = C_s + C_u \cdot N + \sum_{j=1}^{L} TC_j^* \phi(j+A, N). \tag{2.1}$$

To apply incremental analysis, we now take the first difference

$$\begin{aligned} \Delta TC_{N,L} &= TC_{N+1,L} - TC_{N,L} \\ &= C_u + \sum_{j=1}^{L} TC_j^* [\phi(j+A+1, N+1) - \phi(j+A, N)]. \end{aligned} \tag{2.2}$$

From (1.1), we have

$$\begin{aligned} \phi(x+1, N+1) - \phi(x, N) &= \alpha q p^{x+1} \left[\binom{N}{x+1} q^{N-x-1} + \cdots + \binom{x+2}{x+1} \bar{\alpha}^{N-x-2} q + \bar{\alpha}^{N-x-1} \right] \\ &\quad - \alpha q p^{x} \left[\binom{N}{x} q^{N-x} + \cdots + \binom{x+1}{x} \bar{\alpha}^{N-x-1} q + \bar{\alpha}^{N-x} \right] \\ &= q[\phi(x+1, N) - \phi(x, N)]. \end{aligned} \tag{2.3}$$

Using (2.3) to simplify (2.2), we have

$$\Delta TC_{N,L} = C_u - q \cdot TC_1^* \phi(A+1, N) - q \sum_{j=2}^{L} (TC_j^* - TC_{j-1}^*) \phi(A+j, N). \tag{2.4}$$

We shall now evaluate the difference terms of the type $(TC_j^* - TC_{j-1}^*)$. Let

$Z(j)$ = starting quantity such that the expected number of non-defectives is equal to j.

Then $Z(j)$ is obtained from the equation

$$Z(j)-j=Z(j)\cdot p-p\frac{\bar{\alpha}}{\alpha}(1-\bar{\alpha}^{Z(j)})$$

or, equivalently,

$$Z(j)=\frac{1}{q}\left[j-p(\bar{\alpha}+\bar{\alpha}^2+\ldots+(\bar{\alpha})^{Z(j)}\right]. \tag{2.5}$$

We assume that the fixed setup cost is much larger than the direct unit cost, i.e.,

$$C_s \gg C_u. \tag{2.6}$$

In order to keep the cost expression reasonably simple, we postulate that the cost of a supplementary run could be based on starting a lot just large enough on the average to satisfy the shortage. If (2.6) holds, this approximation is not too serious. Thus,

$$TC_j^* \cong C_s+Z(j)\cdot C_u \tag{2.7}$$

and

$$TC_j^*-TC_{j-1}^* \cong [Z(j)-Z(j-1)]\cdot C_u. \tag{2.8}$$

It is worthwhile to point out here that (2.8) should be an adequate approximation even in those cases where (2.6) does not strictly hold. Using (2.5) in (2.8), we obtain

$$TC_j^*-TC_{j-1}^*=\frac{C_u}{q}(1-p\cdot(\bar{\alpha})^{Z(j)}) \cong \frac{C_u}{q}. \tag{2.9}$$

Furthermore,

$$TC_1^*=C_s+\frac{C_u}{q}. \tag{2.10}$$

LEMMA 1: If $\alpha \geq \frac{p}{1+p}$, then $\phi(x, N)$ is a monotone decreasing function of x for $x \geq \bar{x}(N)+1$.

PROOF:

$$\begin{aligned}\bar{x}(N)+1&=N\cdot p-p\frac{\bar{\alpha}}{\alpha}(1-\alpha^N)+1\\ &\geq N\cdot p-p\frac{\bar{\alpha}}{\alpha}+1\\ &=N\cdot p-\frac{p}{\alpha}+p+1\\ &\geq N\cdot p.\end{aligned}$$

Let $B_p(\cdot, N)$ be the binomial probability mass function with parameters p and N. It is well known that $B_p(x, N)$ is monotone decreasing for $x \geq N\cdot p$. It follows, therefore, that $B_p(x, N')$ is monotone decreasing for $x \geq N\cdot p$ for all $N' \leq N$.

From (1.1), we obtain

$$\phi(x, N)=\alpha\cdot B_p(x, N)+\bar{\alpha}\cdot\alpha\cdot B_p(x, N-1)+\ldots+(\bar{\alpha})^{N-x}\cdot\alpha\cdot B_p(x, x). \tag{2.11}$$

Since each term on the right hand side of (2.11) is monotone decreasing for $x \geq N \cdot p$, we conclude that $\phi(x, N)$ is monotone decreasing for $x \geq \bar{x}(N) + 1$. This completes the proof.

DEFINITION 1: A real valued function $f(\cdot)$ on E^1 is called *unimodal* if $f(\cdot)$ is monotone on its domain or if, for some x_0, $f(\cdot)$ is non-decreasing on $(-\infty, x_0]$ and non-increasing on $[x_0, \infty)$. Thus, $f(x_0)$ is the global maximum of the function $f(x_0)$ and x_0 is the optimizing value of the independent variable.

DEFINITION 2: Suppose that $f(\cdot)$ is a unimodal function on E^1. Then the point x^* in its domain is the *smallest optimizing value* of the independent variable if, either $f(x^*) > f(x)$ for all $x < x^*$ and $f(x^*) \geq f(x)$ for all $x > x^*$, or x^* is the right end point of the domain and $f(\cdot)$ is a monotone strictly increasing function.

THEOREM 1: If $C_s \geq C_u$ and $\alpha \geq p/1+p$, then $-TC_{N,L}$ is a unimodal function of N for $N \geq L + x(N)$. Furthermore, there exists a smallest optimum starting quantity N^* where N^* is the smallest integer $(\geq L + \bar{x}(N^*))$ such that

$$\frac{C_u}{q} \frac{Pr(D \leq A^* | N^*)}{\phi(A^*+1, N^*)} \geq C_s, \tag{2.12}$$

where

$Pr(D \leq A^* \mid N^*) =$ probability that the number of defectives is less than or equal to $(N^* - L)$ given that N^* items are processed.

PROOF: Using (2.9) and (2.10) in (2.4), we obtain

$$\frac{\Delta TC_{N,L}}{q \cdot \phi(A+1, N)} = \frac{C_u}{q} \frac{Pr(D \leq A \mid N)}{\phi(A+1, N)} - C_s. \tag{2.13}$$

Let

$$M(N, L) = \frac{Pr(D \leq A \mid N)}{\phi(A+1, N)} \text{ for } N \geq L. \tag{2.14}$$

Now $M(N, L)$ is a monotone increasing function of N if and only if

$$\frac{Pr(D \leq A \mid N)}{\phi(A+1, N)} \leq \frac{Pr(D \leq A+1 \mid N+1)}{\phi(A+2, N+1)} \text{ for } N > L. \tag{2.15}$$

We have

$$\begin{aligned} \phi(0, N+1) &= \alpha[q^{N+1} + \bar{\alpha}q + \ldots + \bar{\alpha}^N q] + \bar{\alpha}^{N+1} \\ &= q\phi(0, N) + p\bar{\alpha}^{N+1} \end{aligned} \tag{2.16}$$

and

$$\phi(x+1, N+1) = q\phi(x+1, N) + p\phi(x, N), \text{ if } 0 \leq x \leq N-1. \tag{2.17}$$

Using (2.16) and (2.17) in (2.15), we conclude that $M(N, L)$ is monotone increasing if and only if

$$\frac{Pr(d \leq A|N)}{\phi(A+1, N)} \leq \frac{qPr(D \leq A+1|N) + pPr(D \leq A|N) + p\bar{\alpha}^{N+1}}{q\phi(A+2, N) + p\phi(A+1, N)}$$

or, equivalently,

$$qPr(D \leq A|N)\phi(A+2, N) \leq qPr(D \leq A+1|N)\phi(A+1, N) + \phi(A+1, N)p\bar{\alpha}^{N+1}$$

or, equivalently,

$$Pr(D \leq A|N)\,[\phi(A+2, N) - \phi(A+1, N)] \leq \left[\phi(A+1, N) + \frac{p}{q}\,\bar{\alpha}^{N+1}\right]\phi(A+1, N). \tag{2.18}$$

Since the right-hand side of (2.18) is positive and $\phi(x, N)$ is non-increasing for $x \geq \bar{x}(N)+1$, the inequality is always true for $N \geq L+\bar{x}(N)$ and it follows that $M(N, L)$ is monotone increasing for $N \geq L+\bar{x}(N)$. Therefore, $\Delta TC_{N,L}$ is also monotone increasing which further implies that $-TC_{N,L}$ is unimodal for $N \geq L+\bar{x}(N)$. Also since

$$\lim_{N\to\infty} \Delta TC_{N,L} = \infty,$$

it follows that $TC_{N,L}$ attains its minimum at some $N \geq L+\bar{x}(N)$. Let N^* be the smallest integer $(\geq L+x(N^*))$ such that

$$\Delta TC_{N,L} \geq 0$$

or, equivalently,

$$\left[\frac{C_u}{q}\right]\frac{Pr(D \leq A|N)}{\phi(A+1, N)} \geq C_s,$$

which is the same as (2.12). Then N^* is the smallest optimum starting quantity, which completes the proof.

NOTE: It is worthwhile to point out here that $\alpha \geq p/1+p$ is only a sufficient (and *not* a necessary) condition for Lemma 1 and Theorem 1. Although we have not been able to prove Lemma 1 for $0 < \alpha < p/1+p$ mathematically (it is trivially true for $\alpha = 0$), we have run several cases on a computer to assure ourselves that it holds for $0 < \alpha < p/1+p$. It follows, therefore, that results in Theorem 1 could be applied for all values of α.

Notice that if the binomial parameter p for state 2 and the order size L are large, then a large number of defectives are likely to be produced if the production run consists of one big lot without any intermediate inspection to determine the state of the production system. The corresponding reject allowance and the total expected cost of satisfying the order will also be large. This suggests that we might be able to do better by allowing periodic process inspections so that if the process is found to have deteriorated to state 2, it could be reset and returned to state 1.

3. OPTIMUM INSPECTION INTERVAL: "ECONOMIC LOT SIZE" APPROACH TO THE REJECT ALLOWANCE PROBLEM.

Consider the reject allowance problem as introduced in section 1 and suppose that it is permitted to inspect the process at any time to determine the state of the production system. Each inspection costs C_i and if the system is found in the "bad state," it can be repaired and transferred to the "good state" at fixed cost C_t. As before, there is an *initial* process setup cost C_s. The items are examined only after the production run is complete and if found short, a makeup run is essential and the setup cost C_s is incurred again. Assume that $C_s > C_t$. The objective is to determine the processing and inspection schedule that minimizes the total expected cost of satisfying the order.

We shall restrict our attention to (r, R) processing-inspection policies. An (r, R) *processing-*

inspection policy is defined as: process R *lots* each of size r in the first production run and inspect the process after processing each lot. Then our objective is to determine the parameters r^* and R^* such that the (r^*, R^*) processing-inspection policy minimizes the total expected cost of satisfying the order* among all such policies and is, therefore, termed an *optimal processing-inspection policy*. Here r^* is called the "optimal inspection interval" or the "economic lot size." As we shall see, r^* depends only on the process parameters while R^* is a non-decreasing function of the order size L.

Notice that after each inspection the production process starts in the "good state." We are interested in the inspection span, r^*, which yields the minimum expected average cost per non-defective produced. Let

$$(3.1)\qquad AC(r) = \text{average cost per non-defective produced when inspection span is } r$$
$$= \frac{\text{total expected cost for a cycle of length } r}{\text{expected number of non-defectives produced during the cycle}}$$
$$= \frac{r \cdot C_u + C_i + (1-\bar{\alpha}^r)C_t}{\frac{\bar{\alpha}}{\alpha}\, q(1-\bar{\alpha}^r)}.$$

Although r must be an integer, since $AC(r)$ is a continuous function of r for $r > 0$, we shall treat r as a continuous variable and find $\bar{r}$ which minimizes $AC(r)$ over $r > 0$. Let $[\bar{r}]$ = largest integer $\leq \bar{r}$. We now prove the following result about the optimality of periodic process inspections.

THEOREM 2:

(i) If $C_t \geq p\bar{\alpha}/q\alpha C_u$, then it is never optimal to inspect the process and Theorem 1 can be used to find N^*, the optimal starting quantity.

(ii) If $C_t < p\bar{\alpha}/q\alpha C_u$ and

$$(3.2)\qquad \frac{p\bar{\alpha}}{\alpha} C_u - qC_t \geq \frac{q + p\bar{\alpha}^2}{\alpha^2} C_i,$$

then $r^* = 1$, i.e., it is optimal to inspect the process after each item is processed.

(iii) If $C_t < p\bar{\alpha}/q\alpha C_u$, (3.2) does not hold and

$$(3.3)\qquad \frac{p\bar{\alpha}}{\alpha} C_u - qC_t > qC_i,$$

then $\bar{r}$ is the solution of the equation,

$$(3.4)\qquad p\,\frac{\bar{\alpha}}{\alpha} C_u - qC_t = C_i\,\frac{q + p\bar{\alpha}^{r+1}}{1 - \bar{\alpha}^r - r\alpha\bar{\alpha}^r},$$

and we take

$$r^* = \begin{cases} [\bar{r}], \text{ if } AC([\bar{r}]) \leq AC([\bar{r}]+1) \\ [\bar{r}]+1, \text{ if } AC([\bar{r}]) > AC([\bar{r}]+1), \end{cases}$$

i.e., it is optimal to inspect the process after each r^* items are processed.

(iv) If (3.3) does not hold, then it is never optimal to inspect the process and Theorem 1 can be used to find N^*, the optimal starting quantity.

*This includes the cost of supplementary runs if needed.

PROOF: From (3.1)

$$\frac{dAC(r)}{dr}=\frac{(C_u-\bar{\alpha}^r \ln \bar{\alpha}\cdot C_t)\left[rq+p\,\frac{\bar{\alpha}}{\alpha}\,(1-\bar{\alpha}^r)\right]-\left(q-p\,\frac{\bar{\alpha}}{\alpha}\bar{\alpha}^r \ln \bar{\alpha}\right)[rC_u+C_i+(1-\bar{\alpha}^r)C_t]}{\left[rq+p\,\frac{\bar{\alpha}}{\alpha}\,(1-\bar{\alpha}^r)\right]^2}. \tag{3.5}$$

The numerator on the right-hand side of (3.5) is

$$Y(r)=\left(p\,\frac{\bar{\alpha}}{\alpha}\,C_u-qC_t\right)(1-\bar{\alpha}^r+r\bar{\alpha}^r \ln \bar{\alpha})-C_i\left(q-p\,\frac{\bar{\alpha}}{\alpha}\,\bar{\alpha}^r \ln \bar{\alpha}\right)$$

so

$$\frac{Y(r)}{q+p\bar{\alpha}^{r+1}}\simeq\left(p\,\frac{\bar{\alpha}}{\alpha}\,C_u-qC_t\right)\frac{1-\bar{\alpha}^r-r\alpha\bar{\alpha}^r}{q+p\bar{\alpha}^{r+1}}-C_i, \tag{3.6}$$

assuming $\ln \bar{\alpha}\simeq-\alpha$.

Let $Z(r)=\dfrac{1-\bar{\alpha}^r-r\alpha\bar{\alpha}^r}{q+p\bar{\alpha}^{r+1}}$.

Then

$$\frac{dZ(r)}{dr}\simeq\frac{qr\alpha^2\bar{\alpha}^r+p\alpha\bar{\alpha}^{r+1}(1-\bar{\alpha}^r)}{(q+p\bar{\alpha}^{r+1})^2} \tag{3.7}$$

$$\geqslant 0 \text{ for } r>0.$$

Also,

$$Z(0)=0, \tag{3.8}$$

$$Z(1)=\frac{\alpha^2}{q+p\bar{\alpha}^2}>0,$$

and

$$\lim_{r\to\infty} Z(r)=\frac{1}{q}. \tag{3.9}$$

Thus, $Z(r)$ is monotone increasing and positive for all $r\geqslant 0$.

(i) If $C_t\geqslant p\bar{\alpha}/q\alpha C_u$, then $Y(r)\leqslant 0$ for all r. Hence, $dAC(r)/dr\leqslant 0$ for all r, which implies that it is optimal not to have any intermediate inspection. This is the case considered in Section 2, so the optimal starting quantity, N^*, is computed as indicated in Theorem 1.

However, if $C_t<p\bar{\alpha}/q\alpha C_u$, then $Y(r)$ and $dAC(r)/dr$ change sign at most once from negative to positive, implying that $-AC(r)$ is a *unimodal* function of r. The other three results are then easily established as follows.

(ii) If (3.2) holds then, from (3.6) and (3.8), we obtain $Y(1)\geqslant 0$, which implies that $dAC(r)/dr\geqslant 0$ for all $r\geqslant 1$, so $r^*=1$.

(iii) If (3.2) does not hold, then $dAC(r)/dr<0$ for $r=1$. Also, since

$$\lim_{r\to q} Z(r)=\frac{1}{q},$$

and (3.3) holds, it follows that $AC(r)$ attains its minimum for some finite value of $r > 1$. Therefore, $\bar{r}$ exists and it is the solution of the equation

$$p\frac{\bar{\alpha}}{\alpha}C_u - qC_l = C_i\frac{q+p\bar{\alpha}^{r+1}}{1-\bar{\alpha}^r-r\alpha\bar{\alpha}^r},$$

which is the same as (3.4).

(iv) If (3.3) does not hold, then $dAC(r)/dr < 0$ for all $r > 0$ and, therefore, it is optimal never to inspect. Once again, this is the case considered in Section 2 and N^* is computed as indicated in Theorem 1. This completes the proof.

It then follows that for given α, p, C_i, C_t, and C_u, the optimal inspection interval, r^*, is determined. We now proceed to determine the second parameter of the (r^*, R^*) optimal policy. R^* represents the number of lots, each of size r^*, processed in the first production run.

Let μ and σ be the mean and standard deviation respectively of the probability mass function $\phi(\cdot, r^*)$. Then $r^*-\mu=$ expected number of non-defectives per lot.

Suppose that we process R such lots and let Y_R be the total number of defectives obtained. Then for large values of R it follows from the Central Limit Theorem that the random variable Y_R is approximately nornally distributed with mean $R\mu$ and variance $R\sigma^2$. Thus, the random variables

$$Z_R=\frac{Y_R-R\mu}{\sigma\sqrt{R}}$$

and

$$Z_{R+1}=\frac{Y_{R+1}-(R+1)\mu}{\sigma\sqrt{R+1}}$$

are both (approximately) normally distributed with mean 0 and variance 1 and will be denoted by the same random variable Z.

Let $CL \equiv$ total expected cost of one lot of size r^*

$$= r^*C_u + C_i + (1-\bar{\alpha}^{r^*})C_t$$

and $TC_R \equiv$ total expected cost of meeting the order requirements if R lots are processed in the first production run

$$= C_s + R \quad CL + \sum_{j=1}^{L} \text{(probability that the final shortage is } j\text{)}$$

$\times$ (minimum expected cost of a makeup run when shortage is j).

Once again, we assume that the initial setup cost (C_s) is of a higher order of magnitude than the processing, inspection, and repair costs. One concludes, therefore, that the probability of more than one makeup run will be negligible for optimum production policies.

Furthermore, a final shortage (if any) could be made up through a supplementary run of size r^*. Hence, the cost of a makeup run $\cong C_s + CL$. Thus,

$$TC_R = C_s + R\cdot CL + (C_s+CL)Pr(Y_R > R\cdot r^* - L)$$

$$= C_s + R\cdot CL + (C_s+CL)Pr\left(Z_R > \frac{R(r^*-\mu)-L}{\sigma\sqrt{R}}\right)$$

and

$$(3.11)\quad \Delta TC_R = TC_{R+1} - TC_R$$
$$= CL + (C_s + CL)\left[Pr\left(Z_{R+1} > \frac{(R+1)(r^*-\mu)-L}{\sigma\sqrt{R+1}}\right) - Pr\left(Z_R > \frac{R(r^*-\mu)-L}{\sigma\sqrt{R}}\right)\right]$$
$$= CL + (C_s + CL)\left[Pr\left(Z > \frac{\sqrt{R+1}}{\sigma}(r^*-\mu) - \frac{L}{\sigma\sqrt{R+1}}\right) - Pr\left(Z > \frac{\sqrt{R}}{\sigma}(r^*-\mu) - \frac{L}{\sigma\sqrt{R}}\right)\right]$$
$$= CL - (C_s + CL)\left[Pr\left(\frac{R(r^*-\mu)-L}{\sigma\sqrt{R}} \leq Z \leq \frac{(R+1)(r^*-\mu)-L}{\sigma\sqrt{R+1}}\right)\right].$$

Let

$$(3.12)\quad T(R, L) = \frac{R(r^*-\mu)-L}{\sigma\sqrt{R}},\ \underline{R} = \frac{L}{r^*-\mu}.$$

Then for $R \geq \underline{R}$, we have $T(R, L) \geq 0$ and the corresponding reject allowance $A = R(r^*-\mu) - L \geq 0$.

THEOREM 3:—TC_R as given by (3.10) is a unimodal function of R for $R \geq \underline{R}$. Furthermore, it attains its minimum at $R = R^*$, where R^* is the smallest integer $(\geq \underline{R})$ such that

$$(3.13)\quad Pr(T(R, L) \leq Z \leq T(R+1, L)) \leq \frac{CL}{C_s + CL}.$$

PROOF: Let

$$(3.14)\quad I(R, L) = \Delta_1 T(R, L) = T(R+1, L) - T(R, L).$$

Then

$$I(R, L) > 0 \qquad \text{for all } R \geq 0$$

and

$$\Delta_1 I(R, L) = I(R+1, L) - I(R, L)$$
$$= T(R+2, L) - 2T(R+1, L) + T(R, L)$$
$$= \frac{r^*-\mu}{\sigma}\left[\sqrt{R+2} - 2\sqrt{R+1} + \sqrt{R}\right] + \frac{L}{\sigma}\left[\frac{2}{\sqrt{R+1}} - \frac{1}{\sqrt{R+2}} - \frac{1}{\sqrt{R}}\right]$$
$$< 0 \qquad \text{for all } R \geq 0.$$

Hence, $T(R, L)$ and $-I(R, L)$ are monotone increasing functions of R for $R \geq \underline{R}$. Since Z is normally distributed with mean 0 and variance 1, it follows that

$$(3.15)\quad M(R) = Pr(T(R, L) \leq Z \leq T(R+1, L))$$

is a monotone decreasing function of R for $R \geq \underline{R}$. Hence, from (3.11), ΔTC_R is a monotone increasing function of R which implies that $-TC_R$ is unimodal for $R \geq \underline{R}$. Furthermore, since

$$\lim_{R \to \infty} \Delta TC_R = \infty,$$

there exists an integer R^* such that TC_{R^*} is the global minimum of TC_R. Also, R^* is the smallest integer $(\geq \underline{R})$ *such that* $\Delta TC_R \geq 0$, i.e.,

$$Pr(T(R, L) \leq Z \leq T(R+1, L)) \leq \frac{CL}{C_s + CL},$$

which completes the proof.

Thus, (3.4) and (3.13) can be used to compute the parameters r^* and R^* of the optimal (r, R) processing-inspection policy and the corresponding reject allowance $A^* = R^* \cdot r^* - L$. We conclude this section after proving the following result.

THEOREM 4: R^* is a monotone increasing function of the order size L.*

PROOF: We have

$$\begin{aligned}\Delta_2 T(R, L) &\equiv T(R, L+1) - T(R, L) \\ &< 0 \qquad \text{for all } R, L > 0\end{aligned}$$

and

$$\begin{aligned}\Delta_2 I(R, L) &\equiv I(R, L+1) - I(R, L) \\ &= T(R+1, L+1) - T(R, L+1) - T(R+1, L) + T(R, L) \\ &= \frac{1}{\sigma}\left[\frac{L+1}{\sqrt{R}} + \frac{L}{\sqrt{R+1}} - \frac{L+1}{\sqrt{R+1}} - \frac{L}{\sqrt{R}}\right] \\ &= \frac{1}{\sigma}\left[\frac{1}{\sqrt{R}} - \frac{1}{\sqrt{R+1}}\right] \\ &> 0 \qquad \text{for all } R, L \geq 0.\end{aligned}$$

Hence, $T(R, L)$ and $-I(R, L)$ are both monotone decreasing functions of L. Therefore, $M(R)$ as defined by (3.15) is a monotone increasing function of L and it follows from (3.13) that R^* is a monotone increasing function of L, as was to be shown.

4. CONCLUSIONS

We have analyzed the reject allowance problem for production processes which undergo stochastic deterioration. We demonstrated that under appropriate conditions a smallest optimum starting run size exists and can be easily computed. We then investigated the optimality of periodic inspections and showed that in several cases one can do better by following an (r, R) inspection-processing policy. Furthermore, we conjecture that for a single-stage production problem where the production process has *any* given probability mass function for the number of defectives generated, one can proceed in a similar manner to derive conditions for an optimum reject allowance to exist and develop computational procedures. In particular, the case of a two-state stochastically deteriorating production system with $p_1 = p$ and $p_2 = 1$ is treated in Ref. [8].

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*However, R^* is not necessarily a monotone *strictly* increasing function of L.

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OPTIMAL SPARES FOR STOCHASTICALLY FAILING EQUIPMENT

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ABSTRACT

A mathematical model is formulated for determining the number of spare components to purchase when components stochastically fail according to a known life distribution function and there is a cost incurred when a component is replaced. Bounds are determined for the optimal inventory which indicate that the inclusion of the replacement cost lowers the optimal inventory.

Since these bounds are no easier to calculate than the optimal spares level, the theory is specialized to components with exponentially distributed time to failure. Procedures are given for calculating the optimal spares level, and numerical examples are provided.

INTRODUCTION

A decision maker requires the use of a piece of equipment over a planning horizon of length, T. The equipment has a component which fails according to a known life distribution, F. The purchase price for the components is c per unit. There is a holding cost of h per component per unit time. A penalty cost of p per unit time is incurred if the equipment is not available for use. And it costs r to replace a failed component with a spare. The decision maker desires to know how many spare components to purchase at the beginning of the planning horizon to minimize his expected costs.

Much research has been done in the mathematical theory of inventory, Scarf [5] and Veinott [6], and on the derivation of maintenance policies, Barlow and Proschan [1] and McCall [3]. Most of the literature has not dealt with the problem that inventories of spare parts are kept to support the implementation of maintenance policies. Notable exceptions are Proschan [4], Beckmann [2], and Wiggins [7]. Proschan finds the optimal spares required to maximize the probability of equipment availability subject to budget constraints. As a special case, Beckmann includes inventory demand distributions which arise from stochastically failing components, and although his objective is to minimize cost, he does not consider replacement costs.

Wiggins considers a model in which the inventory is limited to a single spare component. An ordering policy for a new spare is determined which minimizes the sum of the costs of waiting for the arrival of a new spare and of holding inventory.

INVENTORY MODELS

The model for the above problem differs from previous inventory models in two ways; the first is by the inclusion of replacement costs, and the second is the structure which results from the specialized demand process. In most inventory models the demand is independent of the amount of inventory held. In an inventory-maintenance system, the demand depends upon the initial inventory level. Since a demand is created at each component failure, when the last spare component fails a demand is created which cannot be satisfied. It is of interest to determine the effect of this structure upon the inventory level.

Let x denote the number of spare components purchased, and Y_i be the random variable, time to failure of the $i-1^{st}$ spare after it is placed into operation. Y_1 is the random variable, time to failure of the original component installed in the equipment. We assume that the Y_i, $i=1,2,\ldots$, are nonnegative, independent, identically distributed random variables, each with distribution function F. We denote by N(t), $t \geq 0$, the renewal process associated with the sequence of random variables, $\{Y_i\}$. N(t) is the number of component failures occurring in the interval [0,t] if an unlimited number of spares are available. Let G(x,T) be the random variable, total cost when x spare components are ordered as inventory to support the single equipment over a period of length, T. Thus, EG(x,T) is the expected total cost. We can express G(x,T) as

$$G(x, T) = cx + h \sum_{k=1}^{x} \min \{S_k, T\} + p(T - \min \{S_{x+1}, T\}) + r \min \{x, N(T)\} \tag{1}$$

where

$$S_k = \sum_{i=1}^{k} Y_i,$$

and c, r, p, and h are the nonnegative cost parameters previously described.

We want to find x^* such that

$$C(T) = \min_{x \geq 0} EG(x, T) = EG(x^*, T). \tag{2}$$

Define $\Delta G(x, T) = G(x+1, T) - G(x, T)$. We have that

$$\Delta G(x, T) = c + h \min \{S_{x+1}, T\} - p \min \{S_{x+2}, T\} + p \min \{S_{x+1}, T\} + r \cdot \begin{pmatrix} 1 \text{ if } N(T) \geq x+1 \\ 0 \text{ if } N(T) < x+1 \end{pmatrix}.$$

and

$$\Delta EG(x, T) = E\Delta G(x, T)$$

$$= c + (h+p)(T(1 - F_{x+1}(T)) + \int_0^T t dF_{x+1}(t)) - p(T(1 - F_{x+2}(T)) + \int_0^T t dF_{x+2}(t)) + rF_{x+1}(T),$$

since $P\{N(T) \geq x+1\} = F_{x+1}(T)$, where $F_k(\cdot)$ denotes the k-fold convolution of $F(\cdot)$. Because $\int_0^T t dF(t) = \int_0^T (1 - F(t)) dt - T(1 - F(T))$, we have

$$\Delta EG(x, T) = c + h \int_0^T (1 - F_{x+1}(t)) dt - p \int_0^T (F_{x+1}(t) - F_{x+2}(t)) dt + rF_{x+1}(T). \tag{3}$$

The following lemmas and corollary are necessary for characterizing the optimal inventory x^*.

LEMMA 1: If Y is a nonnegative random variable with distribution function $F(\cdot)$, then

$$\int_0^T (F_{k-1}(t) - F_k(t)) dt \geq \int_0^T (F_k(t) - F_{k+1}(t)) dt.$$

PROOF:

$$\int_0^T (F_k(t) - F_{k+1}(t))dt = \int_0^T \int_0^t (F_{k-1}(t-u) - F_k(t-u))dF(u)dt$$

Making the transformation $y = t-u$, $x = u$, and noting that the Jacobian is one, we get

$$\int_0^T (F_k(t) - F_{k+1}(t))dt = \int_0^T \int_0^{T-y} (F_{k-1}(y) - F_k(y))dF(x)dy$$

$$= \int_0^T (F_{k-1}(y) - F_k(y))F(T-y)dy$$

$$\leqslant \int_0^T (F_{k-1}(y) - F_k(y))dy. \| |$$

LEMMA 2: If Y is a nonnegative random variable with distribution function $F(\cdot)$, then

$$\int_0^T (1-F_n(t))dt \geqslant n \int_0^T (F_{n-1}(t) - F_n(t))dt.$$

PROOF:

$$\int_0^T (1-F_n(t))dt = \sum_{k=1}^{n} \int_0^T (F_{n-1}(t) - F_n(t))dt.$$

$$\geqslant n \int_0^T (F_{n-1}(t) - F_n(t))dt \text{ by lemma 1,}$$

where $F_0(t) \equiv \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases}$ $\| |$

COROLLARY 1: If Y is a nonnegative random variable with distribution function $F(\cdot)$ and $EY < \infty$, then

$$\int_T^\infty (1-F_n(t))dt \leqslant n \int_T^\infty (F_{n+k}(t) - F_{n+k+1}(t))dt$$

$$\text{for } k = -1, 0, 1, 2, \ldots, n \geqslant 1.$$

PROOF:

$$\int_0^\infty (1-F_n(t))dt = nEY = n \int_0^\infty (F_{n+k}(t) - F_{n+k+1}(t))dt \tag{4}$$

$$\text{for } k = -n, \ldots, -1, 0, 1, 2, \ldots.$$

By lemma 2 we have

$$\int_0^T (1-F_n(t))dt \geqslant n \int_0^T (F_{n-1}(t) - F_n(t))dt$$

$$\geqslant n \int_0^T (F_{n+k}(t) - F_{n+k+1}(t))dt \tag{5}$$

by lemma 1 for $k = -1, 0, 1, 2, \ldots$.

Thus subtracting (5) from (4) gives the desired result. $\|$ We previously found

$$\Delta EG(x, T) = c + rF_{x+1}(T) + \phi(x, T), \tag{6}$$

where

$$\phi(x, T) = h\int_0^T (1-F_{x+1}(t))dt - p\int_0^T (F_{x+1}(t)-F_{x+2}(t))dt. \tag{7}$$

Since c, r, h, and p are positive,

$$\int_0^T (1-F_{x+1}(t))dt$$

is clearly nondecreasing in x, and by lemma 1

$$-\int^T (F_{x+1}(t)-F_{x+2}(t))dt$$

is nondecreasing in x, then we see that $\phi(x, T)$ is nondecreasing in x.
Define

$$U(x, T) = c + r + \phi(x, T)$$

and

$$L(x, T) = c + \phi(x, T).$$

From (6) and $0 \leq F_{x+1}(T) \leq 1$ we have

$$L(x, T) \leq \Delta EG(x, T) \leq U(x, T).$$

Let $\underline{x}$ be the largest integer such that $U(\underline{x}, T) \leq 0$ and $\bar{x}$ be the smallest integer such that $L(\bar{x}, T) \geq 0$.

LEMMA 3: If x^* is optimal, then $\underline{x} \leq x^* \leq \bar{x}$.

PROOF: $L(x, T)$ and $U(x, T)$ are nondecreasing in x since $\phi(x, T)$ is nondecreasing. If $x > \bar{x}$, then $\Delta EG(x, T) \geq L(x, T) \geq L(\bar{x}, T) \geq 0$ and $EG(x, T)$ is an increasing function of $x > \bar{x}$.

If $x < \underline{x}$, then $\Delta EG(x, T) \leq U(x, T) \leq U(\underline{x}, T) \leq 0$ and $EG(x, T)$ is a decreasing function of $x < \underline{x}$. Thus $\underline{x} \leq x^* \leq \bar{x}$. ||

The bounds $\underline{x}$ and $\bar{x}$ are the sharpest available. It should be obvious that if $r = 0$, i.e., we do not consider replacement costs, then $\Delta EG(x, T) = L(x, T)$. Thus $\bar{x}$ is optimal for this problem. Lemma 3 shows us that the inclusion of replacement costs lowers the optimal inventory.

In general $\bar{x}$, x^*, and $\underline{x}$ are difficult to calculate. The following lemma provides an easily calculable upper bound which is looser than $\bar{x}$.

LEMMA 4:
$$\bar{x} \leq \frac{p-h}{h}.$$

PROOF: From Eq. (7),

$$\phi(x, T) = h\left\{\int_0^T (1-F_{x+1}(t))dt - \frac{p}{h}\int_0^T (F_{x+1}(t)-F_{x+2}(t))dt\right\}.$$

Combining lemmas 1 and 2, we get

$$\int_0^T (1-F_{x+1}(t))dt - (x+1)\int_0^T (F_{x+1}(t)-F_{x+2}(t))dt \geq 0.$$

Thus for $p/h \leq x+1$, $\phi(x, T) \geq 0$. Since $c > 0$ we have $L(x, T) > \phi(x, T) \geq 0$ for $x \geq p-h/h$, so $\bar{x} \leq p-h/h$.

This upper bound results from the special structure of the inventory model. Since the cost, p, can be interpreted as the value of using the equipment, this bound implies that $\bar{x}=0$ unless the inventory carrying cost per unit time is strictly less than the value per unit time of using the equipment.

It is not possible to predict the effect of changes in the cost parameters upon the optimal inventory level because $\Delta EG(x, T)$ is not monotone in x. But we can obtain some intuition by seeing how the bounds $\bar{x}$ and $\underline{x}$ change.

LEMMA 5: If $\underline{x}(\pi)$ and $\bar{x}(\pi)$ are the lower and upper bounds for x^* as functions of a parameter π then

(i.) $\underline{x}(\cdot), \bar{x}(\cdot)$ are nonincreasing in c and h,

(ii.) $\underline{x}(\cdot), \bar{x}(\cdot)$ are nondecreasing in p,

(iii.) $\underline{x}(\cdot)$ is nonincreasing and $\bar{x}(\cdot)$ is constant in r.

PROOF: We prove only (iii.) since the method is identical for (i.) and (ii.). Let $r > r'$, then $\underline{x}(r)$ is the largest integer such that $U(\underline{x}(r), T) = c+r+\phi(\underline{x}(r), T) \leq 0$. But for all $x \geq 0$, $c+r+\phi(x, T) < c+r'+\phi(x, T)$. Therefore, $\underline{x}(r') \leq \underline{x}(r)$ since $U(x, T)$ is nondecreasing in x and $\underline{x}(r')$ is the largest integer such that $c+r'+\phi(\underline{x}(r'), T) \leq 0$.
$L(\bar{x}(r), T) = c+\phi(\bar{x}(r), T) = c+\phi(\bar{x}(r'), T) = L(\bar{x}(r'), T)$ and $\bar{x}(r) = \bar{x}(r')$.| |

The following Lemma provides a sufficient condition for ordering spares.

LEMMA 6: If

$$p \int_0^T [F(t) - F_2(t)]\,dt \geq c+r+hEY$$

then $x^* > 0$.

PROOF: We have

$$\int_0^T (1-F_1(t))\,dt < EY$$

and $F(T) \leq 1$ so

$$\Delta EG(0, T) = c+rF(T)+h\int_0^T (1-F(t))\,dt - p\int_0^T (F(t)-F_2(t))\,dt$$

$$\leq c+r+hEY-p\int_0^T (F(t)-F_2(t))\,dt \leq 0.| |$$

The difficulty in calculating $\underline{x}$, x^*, and $\bar{x}$ arises from convoluting the distribution function an unknown number of times. One has no choice except to start at the origin and successively calculate $\Delta EG(x, T)$ for $x=0, 1, 2, \ldots$ until one reaches x^* such that $\Delta EG(x^*-1, T) \leq 0$ and $\Delta EG(x^*, T) > 0$.

APPLICATION EXPONENTIAL DISTRIBUTION

It has been shown, Barlow and Proschan [1], that one need not consider preventive maintenance when the hazard function, $h(t) = f(t)/1-F(t)$, is constant or nonincreasing.

Thus our model is applicable for life distributions which have hazard functions satisfying these properties. We explore in detail the exponential distribution, which is the case of a constant hazard function.

Thus $F(t)=1-e^{-\lambda t}$ where $1/\lambda>0$ is the mean time between failures. $F_n(t)$ is the gamma distribution with parameters n and λ, and is given by:

$$F_n(t)=1-e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!};\ n=1, 2, \ldots;\ \lambda>0.$$

Thus

$$\begin{aligned}\int_0^T (1-F_{x+1}(t))\,dt &= \int_0^T e^{-\lambda t}\sum_{n=0}^{x}\frac{(\lambda t)^n}{n!}\,dt\\ &= \sum_{n=0}^{x}\left[\frac{1}{\lambda}-\frac{1}{\lambda}e^{-\lambda T}\sum_{k=0}^{x}\frac{(\lambda T)^k}{k!}\right]\\ &= \frac{1}{\lambda}\left(x+1-e^{-\lambda T}\sum_{n=0}^{x}(x+1-n)\frac{(\lambda T)^n}{n!}\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^T [F_{x+1}(t)-F_{x+2}(t)]\,dt &= \int_0^T \frac{(\lambda t)^{x+1}}{(x+1)!}e^{-\lambda t}dt\\ &= \frac{1}{\lambda}-\frac{1}{\lambda}e^{-\lambda T}\sum_{n=0}^{x+1}\frac{(\lambda T)^n}{n!}.\end{aligned}$$

Normalizing on the holding cost; i.e., $p'=p/h$, $c'=c/h$, $r'=r/h$, and letting $\tau=\lambda T$ (the expected number of failures), we have

$$\begin{aligned}EG(x,\tau) &= c'+r'\left(1-e^{-\tau}\sum_{n=0}^{x}\frac{\tau^n}{n!}\right)\\ &\quad+\frac{1}{\lambda}\left(x+1-e^{-\tau}\sum_{n=0}^{x}(x+1-n)\frac{\tau^n}{n!}\right)\\ &\quad-\frac{p'}{\lambda}\left(1-e^{-\tau}\sum_{n=0}^{x+1}\frac{\tau^n}{n!}\right),\end{aligned}\tag{8}$$

which can be reduced to

$$\begin{aligned}\Delta EG(x,\tau) &= \Delta EG(x-1,\tau)+\frac{p'}{\lambda}e^{-\tau}\frac{\tau^{x+1}}{(x+1)!}-r'e^{-\tau}\frac{\tau^x}{x!}\\ &\quad+\frac{1}{\lambda}\left(1-e^{-\tau}\sum_{n=0}^{x}\frac{\tau^n}{n!}\right),\ \text{for } x=1, 2, \ldots,\end{aligned}\tag{9}$$

and

$$\Delta EG(0,\tau)=c'+r'(1-e^{-\tau})+\frac{1}{\lambda}(1-e^{-\tau})-\frac{p'}{\lambda}(1-e^{-\tau}(1+\tau)).\tag{10}$$

Using Lemma 6, we find that it is optimal to order $x^*>0$ if

$$c'+r'+\frac{1}{\lambda}-\frac{p'}{\lambda}(1-e^{-\tau}(1+\tau))\le 0.$$

The expected total cost of ordering can be obtained from Eq. (1) and is given by:

$$EG(x, T) = EG(x-1, T) + \Delta EG(x-1, T) \tag{11}$$

with

$$\begin{aligned} EG(0, T) &= E\{p(T - \min[y, T])\} \\ &= p\left(T - \frac{1}{\lambda}(1 - e^{-\lambda T})\right), \end{aligned} \tag{12}$$

or

$$EG(0, \tau) = \frac{p}{\lambda}(\tau - (1 - e^{-\tau})) .$$

Whereas Eqs. (9–12) are useful for calculating, they provide little insight into the behavior of the spares order quantity. We can rewrite (8) as follows

$$\Delta EG(x, \tau) = B(x) + e^{-\tau} A(x, \tau) \tag{13}$$

where

$$B(x) = c' + r' + \frac{1}{\lambda}(1 + x) - \frac{p'}{\lambda};$$

and

$$A(x, \tau) = (p'/\lambda)(\tau^{x+1}/(x+1)!) + \sum_{n=0}^{x} (\tau^n/n!)\left(p'/\lambda - r - \frac{(x+1)}{\lambda}\right) + (\tau/\lambda) \sum_{n=0}^{x-1} (\tau^n/n!)$$

It is easy to see that if $B(x) < 0$ then $A(x) > 0$ and that $B(x)$ is strictly increasing in x at rate $1/\lambda$.

Thus $EG(x, \tau) > B(x)$ for $B(x) \leq 0$. So we never need consider any x greater than x', where x' is the greatest integer such that

$$x' \leq p' - 1 - \lambda c' - \lambda r'.$$

We have

$$\Delta A(x, \tau) = (p'/\lambda)(\tau^{x+2}/(x+2)!) - r'(\tau^{x+1}/(x+1)!) - \frac{1}{\lambda} \sum_{n=0}^{x+1} (\tau^n/n!). \tag{14}$$

Thus $\lim_{x \to \infty} e^{-\tau} A(x, \tau) = -1/\lambda$ and for x sufficiently large $\Delta EG(x, \tau)$ approaches a positive constant. In many cases the convergence is quite rapid as illustrated in the numerical examples (Figures 1 and 2, Table 1).

From Figure 1 it appears that the optimal ordering policy $x^*(\tau)$ is nondecreasing in τ and that there exists a τ' such that $x^*(\tau) = x'$ for all $\tau \geq \tau'$. We were not able to obtain a formal proof of this because

$$\frac{\partial}{\partial \tau} e^{-\tau} A(x, \tau) = e^{-\tau}(-\Delta A(x, \tau))$$

and from Eq. (14) $\Delta A(x, \tau)$ is neither monotone in x nor has the same sign for all x's.

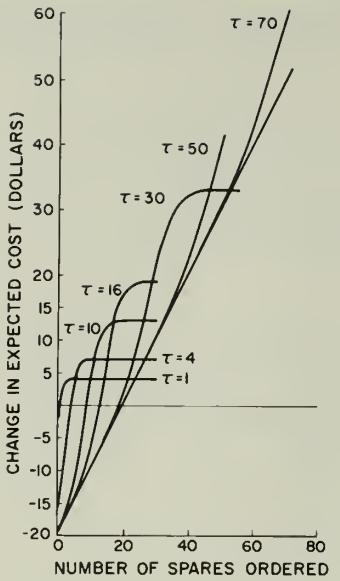

FIGURE 1. Graphs of $\Delta EG(x, \tau)$ *vs* x for the numerical example $\lambda = 1$, $p' = 25$, $c' = 3$, and $r' = 2$

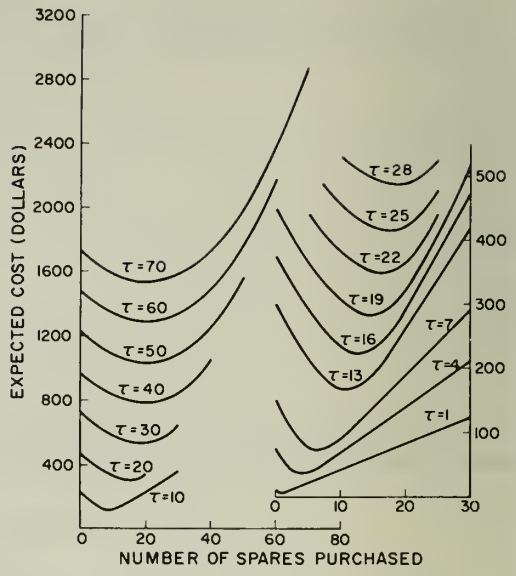

FIGURE 2. Graphs of $EG(x, \tau)$ *vs* x for the numerical example $\lambda = 1$, $p' = 25$, $c' = 3$, and $r' = 2$

TABLE 1

Numerical Example: $p' = 25$, $c' = 5$, $r' = 0.5$, $\lambda = 0.5$ and $\tau = 9$

x	$A(x, \tau)$	$B(x)$	$\Delta EG(x, \tau)$
0	497.50	−42.500	−42.438
1	2498.00	−40.500	−40.191
2	8451.75	−38.500	−37.456
3	21714.75	−36.500	−33.820
4	45292.06	−34.500	−28.910
5	80076.75	−32.500	−22.617
6	123806.66	−30.500	−15.221
7	171464.38	−28.500	−7.339
8	216927.53	−26.500	.270
9	254917.30	−24.500	6.959
10	282303.58	−22.500	12.339
11	298377.88	−20.500	16.322
12	304300.10	−18.500	19.053
13	302207.35	−16.500	20.795
14	294414.42	−14.500	21.833
15	282914.86	−12.500	22.414
16	269188.76	−10.500	22.720
17	254217.54	−8.500	22.872
18	238594.20	−6.500	22.944
19	222649.44	−4.500	22.977

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A DEMAND PREDICTION TECHNIQUE FOR ITEMS IN MILITARY INVENTORY SYSTEMS*

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ABSTRACT

A brief description of a new general approach to demand prediction problems is given. This is based on the demand characteristics of parent components and the parts installed in these components. A specific model is presented which employs this information. The paper concludes with empirical tests of the model.

INTRODUCTION

Considerable attention has been given to the development of a methodology for predicting demands for individual repair parts in military inventory systems. With few exceptions the results have not been promising. The few exceptions have been in two contexts. One has been for items with relatively high usage rates which represent a very small percent of the total items in a military inventory system. The second context has been the prediction of demands for aggregations of unique items. Neither of these contexts are of concern here. In this paper attention is focused on the problem of predicting demands for the great majority of individual items in military inventory systems which are observed to have extremely low usage rates.**

One important aspect of the Navy inventory system will be considered here as the motivation for improved demand prediction techniques. This is the allowance list or shipboard stocking problem. This problem is defined as the determination of the range and depth of repair parts, i.e., allowance quantities, to be stocked aboard ship to meet uncertain demands. In calculating allowance quantities, a significant input is the repair part usage estimate. A usage estimate, it should be noted, is an estimate of the number of units of a part that will require replacement per unit time period per unit of installed population. A major task is to develop procedures for revising the technician's usage estimate as more experience is gained and recorded by the collection of usage data. This paper is devoted to this objective.

The major obstacle in predicting future demands for repair parts is the extremely low and sporadic usage of these items. The first comprehensive evidence of the very high degree of uncertainty of demand appeared in [4]. Since that time, the finding of very low and sporadic usage has been confirmed in many other studies. A major problem is that a very large number of items deemed wearable are not demanded at all over extended time periods, and most of the few items which are demanded are demanded infrequently and with very small quantities. It is this extreme condition which has in the past prohibited the successful application of previously proposed techniques. This is true for those techniques which

*The authors wish to acknowledge comments by referees leading to improvement of this paper. Thanks are also due to Edward Boback and William Hise for their programming assistance.

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**A summary of repair part demand prediction experiments may be found in [1].

require functional relationships between item usage and operational parameters such as steaming hours, flying hours, and the like, and for those which require only a past history of demands.

The important and different orientation of the procedure described here is that it considers not only line-item (i.e., repair part) demands, but also, the demand characteristics of a line-item's parent component. The principal characteristic of the methodology is to utilize information concerning the relative frequency of hits per component where a hit is defined as a requirement for a repair part in a component; this is to be contrasted with the number of units needed for replacement of a given part. The relative frequency of hits for each component may be used to categorize components as being "users" or "non-users." If a part is installed in the population of components which are "users," it may then be treated differently than some part installed in the population of components which are "non-users."

One characteristic associated with low and sporadic line-item usage is that over time the range of parts used changes markedly. For example, typically the majority of those parts used by a ship during one year are not the same parts used in the next year. A second characteristic is that given two or more identical or nearly identical ships, there is little commonality of parts used. An important postulate for the procedure to be described is that on a component basis the demand behavior is more consistent. Evidence in support of this basis postulate is provided in Table 1.

TABLE 1

Cumulative percent of transactions	Cumulative percent	
	Demanded parts	Components experiencing demands
10.0	1.4	0.1
20.0	3.5	0.4
30.0	6.3	1.0
40.0	10.0	1.9
50.0	15.1	3.5
60.0	22.0	5.9
70.0	31.7	10.3
80.0	46.8	18.0
90.0	71.7	34.4
95.0	85.9	51.9
100.0	100.0	100.0

Table 1 shows the cumulative percent of transactions, for 61 patrols of five submarines, accounted for by a given percent of parts and components. As may be seen, 22.0 percent of the demanded parts accounted for 60.0 percent of the transactions, whereas the same percent of transactions was accounted for by only 5.9 percent of the components experiencing demands for parts. This phenomenon of greater consistency in demand among components can only be attributed in a minimum way to a larger number of repair part-applications being installed in the components experiencing demands for parts.* For example, the previously noted components accounting for 60.0 percent of the transactions contained

*The term part-application is used to refer to a unique part-component combination. For example, if the same part is installed in two different components, this would represent two part-applications.

but 21.1 percent of the part-applications installed in the components with demanded parts. Given the observed demand behavior at the component level, there remains the question as to how to develop an explicit measure of component unreliability and exploit this information in order to revise line-item usage estimates, thereby leading to more efficient allowance lists.

In this paper, we will first present a theoretical model derived from the work of George Rasch [5]. The model is described in some detail as it introduces a way in which "component unreliability" and other factors, such as "patrol severity," can be measured and taken into account. The model is then modified to provide a solution to the particular problem of revising repair part usage estimates. In a final section, an application and evaluation of the model is made by employing submarine usage data.

A PROCEDURE FOR DETERMINING COMPONENT UNRELIABILITY AND PATROL SEVERITY FACTORS

For purposes of this paper it is important to keep in mind the distinction between range or number of different parts demanded and depth or number of units of a particular part demanded. To clarify this distinction further, the reader may think of a set of stock record cards where each card represents a unique item identified by a unique stock number. On each card may appear the quantity of the item used. When reference is made in this paper to the range or number of different parts demanded during some time interval, it pertains in this illustration to the number of different stock numbers used or the number of cards on which usage entries are made. It does not pertain to the quantities entered on the cards. The quantities listed on the cards represent the depth or number of units demanded.

Suppose we have a component which contains a range of N different parts and a probability θ that any of these different parts will be demanded during a patrol, where a demand is defined as a requirement for one or more units of a given part. The assumption of a common value of θ is appropriate in the context of the demand prediction problem, since the probabilities of demand for the various parts are, in almost all cases, extremely small. Assume further that demands for different parts in a single component are independently distributed. The assumption of independence within the component is the one of local independence which is made in many fields including demand prediction. On this basis, the probability of a different parts being demanded during a patrol may be written in accordance with the binominal distribution as

$$P\{a \mid N\} = \binom{N}{a}\theta^a(1-\theta)^{N-a}. \tag{1}$$

In what follows, we will use the Poisson distribution as an approximation to the binomial since, in almost all cases, the values of θ are extremely small. It is of interest to note that the Poisson model is also appropriate if the problem is re-formulated as a time-dependent process.

Let a = the number of different parts demanded for a given component during a patrol.

N = the number of different parts installed in the component.

θ = the probability of a part being demanded for a given component during a patrol.

The probability that there will be a different parts demanded during a patrol is

$$P\{a \mid N\} = e^{-\lambda}\frac{\lambda^a}{a!} \tag{2}$$

where

$$\lambda = N\theta \tag{3}$$

is the expected number of different parts demanded.

We now proceed to develop a structure which will take into account both component unreliability and patrol severity. Let us designate the given component by the subscript v and the particular patrol by the subscript i. We assume that the expected number of different parts demanded in component v, containing a total of N_v different parts, during patrol i, is given by

$$\lambda_{vi} = N_v \theta_{vi} = N_v C_v S_i \tag{4}$$

where C_v denotes the component factor which characterizes each component in terms of its "unreliability," i.e., the degree to which the component is an item user* and S_i denotes the patrol factor.† From (4), it follows that the more unreliable the component or the larger the patrol factor, the larger the value of θ_{vi}. Notice that this model is a multiplicative one rather than the frequently criticized linear model.

Thus far, the discussion has been restricted to the expected range, i.e., expected number, of different repair parts demanded in a particular component during a given patrol. Hence, it should be noted that the first step directed at computing component unreliability factors and patrol factors is based on the range of parts demanded. These factors are defined and measured this way throughout the remaining discussion.

In order to compute component unreliability and patrol factors, a table can be constructed showing the number of different parts actually demanded in component v during patrol i. The observed count of different parts which we designate by a_{vi} is an estimate of λ_{vi} and would be entered in row v and column i of such a table. In doing this, one records the number of hits for different items against each component during each patrol. A numerical example is provided later in Table 3.

Associated with the aforementioned table is another table showing the expected values of the observed a_{vi}, an example of which is provided in Table 2.

TABLE 2

Component v \ Patrol i	1	2	.....	Σ
1	$E(a_{11}) = \lambda_{11} = C_1 N_1 S_1$	$E(a_{12}) = \lambda_{12} = C_1 N_1 S_2$	.....	$E\left(\sum_i a_{1i}\right) = C_1 N_1 \sum_i S_i$
2	$E(a_{21}) = \lambda_{21} = C_2 N_2 S_1$	$E(a_{22}) = \lambda_{22} = C_2 N_2 S_2$	.....	$E\left(\sum_i a_{2i}\right) = C_2 N_2 \sum_i S_i$
3	$E(a_{31}) = \lambda_{31} = C_3 N_3 S_1$	$E(a_{32}) = \lambda_{32} = C_3 N_3 S_2$	.....	$E\left(\sum_i a_{3i}\right) = C_3 N_3 \sum_i S_i$
.....	.....	.....	.....	.....
.....	.....	.....	.....	.....
.....	.....	.....	.....	.....
Σ	$E\left(\sum_v a_{v1}\right) = \left(\sum_v C_v N_v\right) S_1$	$E\left(\sum_v a_{v2}\right) = \left(\sum_v C_v N_v\right) S_2$	.....	$E\left(\sum_v \sum_i a_{vi}\right) = \sum_v C_v N_v \sum_i S_i$

*In this sense, component unreliability refers more to the usage aspects of unreliability than to the maintenance or failure aspects of unreliability, although the latter sense is partly implied by the former.

†The patrol factor may be viewed as a measure of patrol severity or it may be considered to be a representation of a time variable.

From (4), the expected value of a_{vi}, $E(a_{vi})$, equals $\lambda_{vi}=C_vN_vS_i$ and hence the expected value of the sum of the a_{vi} for any row v is $C_vN_v\sum_i S_i$. Likewise the expected value of the sum of the a_{vi} for any column i equals $\sum_v (C_vN_v)S_i$. These relations will be used to obtain estimates of C_v and S_i from the observed data.

It is to be noted that each C_v and S_i in Table 2 cannot be determined uniquely since if the C_v are multiplied by any constant, say k, and the S_i are divided by the same constant, the resulting product is unchanged. Note, however, that the *relative* values of each C_v and S_i can be determined uniquely. In estimating values for these parameters in a given problem, scale constants can be chosen which are applicable to the particular problem being studied.

In the present context we begin with estimating relative patrol factors. A first step for estimating S_i by a sample quantity S_i^* is to assign a value to $\sum_i S_i$, say $\sum_i S_i=s$, where s is an appropriately chosen constant. In this problem, s was equated to the number of observed patrols. By choosing s in this manner a useful property is obtained, namely that the mean of the patrol factors equals one. Once having determined s, an estimate of the individual patrol factors can be obtained from

$$S_i^*=s\sum_v a_{vi}\Big/\sum_v\sum_i a_{vi}. \tag{5}$$

The rationale is as follows. Since

$$E\left(\sum_i a_{vi}\right)=C_vN_v\sum_i S_i=C_vN_vs,$$

we set

$$\sum_i a_{vi}=C_v^*N_vs. \tag{6}$$

Solving (6) for C_v^*, we have

$$C_v^*=\sum_i a_{vi}/N_vs, \tag{7}$$

where all the terms on the right are known quantities. Summing (6) over v, we obtain

$$\sum_v\sum_i a_{vi}=s\sum_v C_v^*N_v,$$

so that

$$\sum_v C_v^*N_v=\sum_v\sum_i a_{vi}/s.$$

Since

$$E\left(\sum_v a_{vi}\right)=\sum_v (C_vN_v)S_i,$$

we set

$$\sum_v a_{vi}=\sum_v (C_v^*N_v)S_i^*$$

from which

$$S_i^*=s\sum_v a_{vi}\Big/\sum_v\sum_i a_{vi}\dagger$$

† Notice that $N_vC_v^*S_i^*$, the estimate of λ_{vi}, equals $\sum_v a_{vi}\sum_i a_{vi}\Big/\sum_v\sum_i a_{vi}$ which is independent of the choice of s.

A PROCEDURE FOR ESTIMATING EXPECTED USAGE OF REPAIR PARTS IN SPECIFIED COMPONENTS DURING A PATROL

In this section we shall augment the model just described in order to obtain revised part-application usage estimates based on observed usage data. For this additional development of the demand prediction model, attention will be directed at estimating the expected number of units demanded for each part j in component v during patrol i. We define this expected value as

$$\delta_{vij} = N_{vj}(rC_vS_i)b_j, \tag{8}$$

where C_v and S_i have the same meaning as before, i.e., measures of component unreliability and the patrol factor, N_{vj} is the number of units installed of the jth part in component v, b_j is a measure of the replacement rate per unit of installed population or "unreliability" of part j, and r is a scale factor needed to adjust for the change in reference from range to depth. It should be noted that in (8), the quantity b_j is modified by C_v and S_i since δ_{vij} pertains to a given part installed in a specific component during a particular patrol.

The present formulation is more complex than the earlier one in that we have changed the focus of the problem by introducing an additional parameter, namely, b_j. What is suggested is that the expected number of units demanded for a part installed in a specific component during a given patrol reflects the "unreliability" of the part, the number of installations of the part in the component, and also the unreliability of the parent component and the patrol factor. These various quantities are multiplied together to estimate the expected number of units demanded for each part-application.

We now turn to the problem of determining N_{vj} and r, and estimating b_j, C_v, and S_i. The quantity N_{vj}, the number of units of part j installed in component v, is obtained from available information describing the component. The estimate of b_j, b_j^*, may be the technician's usage estimate or a quantity computed from the recorded usage data for each part. For example, if

$$n_{ji} = \text{number of units of part } j \text{ demanded over all its applications during patrol } i,$$

b_j^* may be obtained from

$$b_j^* = \Big(\sum_i n_{ji}\Big) \Big/ s \sum_v N_{vj}. \tag{9}$$

Because we are concerned with predicting usage quantities for future patrols, we begin by assuming that all patrols will be of average severity. The representation of this assumption is to choose all $S_i^* = 1$ so that in this case (8) reduces to

$$\delta_{vj} = N_{vj}(rC_v)b_j.^{\dagger} \tag{10}$$

It should be noted, however, that (8) is still the basic statement of the demand prediction model.

We now let C_r^* denote the estimate of rC_r, thus modifying (7) to read

† The subscript i has been dropped since all patrols are assumed to have the same S_i.

$$C_v^* = r \sum_i a_{vi}/N_v s, \tag{11}$$

and require that the choice of r satisfy the constraint

$$\sum_v C_v^* = r \sum_v \left(\sum_i a_{vi}/N_v s \right) = k \tag{12}$$

where k is the number of different components subject to failure.†

The simplicity of this approach is important for practical application of demand prediction models to large scale and complex inventory systems. It will be noticed from (11) that whatever the choice of r, the relative values of C_v^* will be the same as in (7). The specification for r in (12) again has the useful property that an average value of 1 is obtained but this time for C_v^*. If all components were of the same unreliability, an estimate of δ_{vj} would be $N_{vj} b_j^*$. The constraint on r is consistent with this possibility.

The computation of the δ_{vj}^* from the model may be illustrated by the example in Table 3 in which hypothetical values of the a_{vi} are shown for two patrols and two components each with two different parts, i.e., $N_1 = N_2 = 2$.

TABLE 3

Component v \ Patrol i	1	2	Σ
1	1	0	1
2	2	1	3
Σ	3	1	4

Setting $\sum_i S_i = s = 2$, C_v^* is estimated from (11) and the footnote below as

$$C_1^* = r \sum_i a_{1i}/N_1 s = \frac{2}{1/4 + 3/4} \cdot \frac{1}{2 \cdot 2} = \frac{1}{2},$$

$$C_2^* = r \sum_i a_{2i}/N_2 s = \frac{2}{1/4 + 3/4} \cdot \frac{3}{2 \cdot 2} = \frac{3}{2}.$$

The arithmetical steps needed in obtaining C_v^* are very simple. Equally important, the data requirements are minimal. The only data elements needed are N_v, $\sum_i a_{vi}$, the number of patrols, s, and the number of different components, k. In particular, only the row marginal totals of the a_{vi} in Table 3 are required, not the individual a_{vi} entries in the body of the table.

†The choice of r satisfying this constraint is, of course,

$$r = k \Big/ \sum_v \left(\sum_i a_{vi} \Big/ N_v s \right)$$

The computation of b_j^* is straightforward. Assuming that during the two patrol period the total number of units of parts 1 and 2 used is 4 and 2, respectively, and that $N_{11}=4$, $N_{21}=6$, $N_{12}=4$, and $N_{22}=8$, we have from (9)

$$b_1^*=\Big(\sum_i n_{1i}\Big)/s\sum_r N_{r1}=4/(2\cdot 10)=0.200,$$

and

$$b_2^*=\Big(\sum_i n_{2i}\Big)/s\sum_r N_{r2}=2/(2\cdot 12)=0.083.$$

Finally, $\delta_{rj}^*=N_{rj}C_r^*b_j^*$:

$$\delta_{11}^*=0.400,\qquad \delta_{12}^*=0.167,\qquad \delta_{21}^*=1.800,\text{ and}\qquad \delta_{22}^*=1.000.$$

In concluding this section, it is again worth noting that b_j^* is based on the number of units used of a repair part over all of its component applications. On the other hand, C_r^* is based on the number of different repair parts used in a given component. These measurement processes are purposefully differentiated to take account of the range of items demanded in the case of C_r and the depth of demand for given items in the case of b_j.

EMPIRICAL TESTS OF PROCEDURES

The data base used for test and evaluation consisted of observations for 49,682 installed part-applications (representing 34,569 different parts) during 82 patrols. The first 61 patrols were taken to be the historical period for which the parameters of the model were estimated. The later 21 patrols were assumed to be the future or test period.

Using data from the 61 patrols, C_r^* was computed for each component. Results for the part-applications are summarized in Table 4.

TABLE 4

C_r^*	(a) Zero usage	(b) Non-zero usage
	No. of part-applications	No. of part-applications
$0.0000 \le C_r^* \le 0.1000$...	22,088	3,863
$0.1000 < C_r^* \le 1.0000$...	12,934	3,730
$1.0000 < C_r^*$.....	3,676	3,391
Total.....	38,698	10,984

In Table 4, Column (a) represents the distribution of C_r^* in terms of the number of part-applications having no usage *in any* of their applications during the first 61 patrols. In Col. (b) is the number of part-applications having positive usage in at least one of their applications during the 61 patrols. It is of interest to note the large proportion of items with zero usage which serves to illustrate some of the problems discussed earlier in the paper.

A first important question is the acceptability of C_r^* as an index for estimation of relative frequency of demand for part-applications. If the procedure is to be useful, one must expect higher future

demands for part-applications with higher values for C_r^*. Table 5 shows the relationship between C_r^* as computed from the first 61 patrols and the frequency of demanded part-applications (shown in Col. (2) of the table) as observed during the *later* 21 patrols.

TABLE 5

C_r^*		(1) No. of part-applications	(2) No. of part-applications Demanded	(3) Percent of part-applications Demanded $\frac{(2)}{(1)} \times 100$
Zero usage $b_j^* = 0$	$0.0000 \leq C_r^* \leq 0.1000$	22,088	115	0.5
	$0.1000 < C_r^* \leq 1.0000$	12,934	151	1.2
	$1.0000 < C_r^*$	3,676	161	4.4
Non-zero usage $b_j^* > 0$	$0.0000 \leq C_r^* \leq 0.1000$	3,863	45	1.2
	$0.1000 < C_r^* \leq 1.0000$	3,730	371	9.9
	$1.0000 < C_r^*$	3,391	1,640	48.4

From Col. (3) of Table 5, it is clear that there is a marked difference in the percent of part-applications demanded among the different C_r^* values. This difference is also evident for those parts with no usage in the first 61 patrols for which $b_j^* = 0$. This verifies the proposition that the component factor is an important one in the determination of repair part demands.

The second and final question considered here is the impact of revising usage estimates via the model and employing the estimates for stock level determinations. The results discussed below are based on the use of the same inventory model as described in [3] for two different cases. In Case I, δ_{rj}^* is employed as the revised usage estimate; in Case II, b_j^* is utilized. The latter estimate is based only on usage for the part without reference to component data; the former is computed using the model just described wherein the part usage estimate is modified by the component unreliability factor. In both cases, the estimates were based on the first 61 patrols. The purpose of the test is to ascertain whether knowledge of the unreliability of the parent component in which a part is installed, as derived from the model here described, leads to a reduction in the range and depth of repair part shortages.

In Cases I and II, the resulting stock levels were assumed to be the same at the beginning of each of the *later* 21 patrols. Given the stock level for each part-application, supply effectiveness measures were computed for each of the two cases for each patrol. The average effectiveness and its variance over the 21 patrols were then computed.

Table 6 shows the range or number of different part-applications stocked for Cases I and II, and the average range or average number of different part-applications for which shortages were recorded. In Col. (1) of this table, the part-applications are stratified by military essentiality class.* Cols. (2) and (3) portray the number of different part-applications stocked for the two cases. The remaining columns, showing shortages for part-applications stocked and not stocked are self-explanatory.

It is clear from Table 6 that use of δ_{rj}^* results in a smaller range of part-applications stocked with

*The derivation of these essentiality classes is described in [2].

no loss in effectiveness. Also, while not shown, the variance associated with average shortages for the model using δ_{vj}^* was somewhat less than for b_j^* which, from the point of view of reducing the number of patrols with many shortages, is an important property of an allowance list.

TABLE 6

Item essentiality	Range stocked		Average range shortages for part-applications [a]			
			Demanded and not stocked		Demanded and stocked	
	Case I	Case II	Case I	Case II	Case I	Case II
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Highest	201	281	0.14	0.10	2.05	3.10
High	680	938	2.29	2.24	10.43	16.81
Low	2,134	2,998	5.00	6.05	16.57	19.71

[a] Per patrol over all part-applications.

Another type of effectiveness measure frequently used is based on only depth or "units short." This is not as straightforward as the previous range counts due to complications stemming from a mixture of units of issue over different items. For sake of completeness, however, Table 7 portrays these depth measures.

TABLE 7

Item essentiality	Units stocked		Average units short [a]	
	Case I	Case II	Case I	Case II
Highest	439	461	13.10	17.29
High	3,777	1,649	128.48	244.48
Low	3,308	3,682	85.71	94.43

[a] Per patrol over all part-applications.

From Table 7 it may be observed that over all item essentiality classes the use of δ_{vj}^* resulted in greater total depth of items stocked. This is one reason for a better shortage performance. It is interesting to note, however, that even for the two categories of items stocked with less depth, the procedure using δ_{vj}^* did better. While not shown in the table, it is of some importance to note that the variance associated with "units short" for Case I was much less than for Case II. For example, in Case I the standard deviation for high essentiality items was 133.35 as compared with a standard deviation of 221.36 for Case II.

In response to the second question raised above, one must conclude that based on the unusually large sample of available data and the nature of the test, i.e., the use of data not utilized in the estimation of δ_{vj} and b_j, the proposed procedure yields improved results relative to conventional methods of estimating line item demands.

SUMMARY

The motivation for this paper is the perplexing problem of demand prediction for repair parts whose usage is low and sporadic. To make this problem more tractable, a model based on work in mathematical psychology, but modified for the problem at hand, is developed. The application of the model builds on evidence that demands on components are more stable than demands for parts. The model provides an estimate of the unreliability of a component which is then used to modify the replacement rate of individual repair parts.

A complicating factor in predicting demand for repair parts in military inventory systems is that they are generally installed in only a few different components and this, in part, accounts for the low and sporadic usage. The model is especially useful in this regard in that it permits identification of the range of parts that are most likely to be demanded from an extremely large collection of parts with very low failure rates. A second and important feature of the model is that it makes possible the ranking of components in terms of unreliability, an aspect that is especially important from the point of view of redesign and provisioning.

The second part of the paper describes empirical evidences that support the theoretical structure. Major emphasis is given to improved predictability possible through the use of the model. Two very different tests are used and each indicates the utility of the approach. It should be noted that in both tests new data, in contrast to data used in estimating component unreliability coefficients and repair part usage estimates, are used to evaluate the model.

The first test confirms the proposition that the probability of demanding a repair part varies directly with the size of the component unreliability coefficient, and this holds for parts with no previous usage history as well as those with usage history. The second test, in which the effectiveness of the model is measured in terms of its ability to effect a reduction in range and depth of shortages, is more pragmatic. In this test, the procedure of using repair part usage estimates based solely on usage for the part without reference to component data is compared with the alternative of modifying such estimates using component unreliability factors developed by the model. Based on a very large sample of data, the proposed procedure yields improved results relative to conventional methods of estimating line item demands.

It should once again be stressed that the benefits to be realized in utilizing component data are two-fold. The use of such data permits improved demand prediction for parts currently in military inventory systems. It also provides important information for both design engineers and logistics support managers for improving the effectiveness of components to be newly introduced into the inventory system.

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THE DISCRETE MAX-MIN PROBLEM*

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ABSTRACT

The historic max-min problem is examined as a discrete process rather than in its more usual continuous mode. Since the practical application of the max-min model usually involves discrete objects such as ballistic missiles, the discrete formulation of the problem seems quite appropriate. This paper uses an illegal modification to the dynamic programming process to obtain an upper bound to the max-min value. Then a second but legal application of dynamic programming to the minimization part of the problem for a fixed maximizing vector will give a lower bound to the max-min value. Concepts of optimal stopping rules may be applied to indicate when sufficiently near optimal solutions have been obtained.

INTRODUCTION

One class of max-min problems is to find the integer x- and y-vectors, $x \epsilon E^n$ and $y \epsilon E^n$, yielding

$$\max_x \min_y \sum_{i=1}^{n} f_i(x_i, y_i)$$

subject to

$$\sum_{i=1}^{n} x_i = X, x \geq 0, \sum_{i=1}^{n} y_i = Y, y \geq 0,$$

where X and Y are fixed quantities. As Danskin [3] points out, a typical application of such a model is in the attack-defense situation, where $f_i(x_i, y_i)$ represents the residual value of target i if the target is defended by x_i defense units and attacked by y_i offense units. The offense wishes to minimize the total residual value,

$$f(x, y) = \sum_{i=1}^{n} f_i(x_i, y_i),$$

and will allocate the Y offensive units to the attack of the n targets so as to accomplish this aim. The defense, knowing the offense will attack in an optimal fashion, will attempt to construct defenses so as to maximize the sum of the residual values of all targets. Thus, the x-player moves first and then the y-player moves in full knowledge of what the x-player has done.

There are a few minor variants to this problem. For example, we can have

$$\min_x \max_y \sum_{i=1}^{n} g_i(x_i, y_i)$$

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*This paper resulted from work done at Braddock, Dunn, and McDonald, Inc. for the SENTINEL System Evaluation Agency (SENSEA) under Contract No. DAHC60-68-C-0022.

where, for the attack-defense application, $g_i(x_i, y_i)$ represents the damage value at the ith target. This damage value function can usually be found directly from the residual value function and, thus, this is only a minor variant from the one above. A slightly more significant change is obtained by interchanging the order of optimization of the x and y vectors; that is, to find

$$\min_y \max_x \sum_{i=1}^{n} f_i(x_i, y_i),$$

where the residual value is now determined by having the y-player move first and then the x-player move in full knowledge of what the y-player has done. Such a model is currently of interest in ballistic missile defense problems, where the offense will move first with his attack and then, after observing the attack, the defense will develop an optimum intercept strategy to counter the attack. The corresponding damage criterion for this problem is to find

$$\max_y \min_x \sum_{i=1}^{n} g_i(x_i, y_i),$$

where $g_i(x_i, y_i)$ is the damage function.

As can be seen by the symmetry of the model, all four of the above variants are similar in structure. Essentially they have a separable payoff function,

$$f(x, y) = \sum_{i=1}^{n} f_i(x_i, y_i)$$

or

$$g(x, y) = \sum_{i=1}^{n} g_i(x_i, y_i),$$

defined over the cartesian product, $X \circ Y$, of the two finite, discrete, nonnegative sets of vectors given by given by

$$\bar{X} = \left\{ x\colon x \epsilon E^n,\ x \geq 0,\ \sum_{i=1}^{n} x_i = X,\ x_i \text{ an integer for } i = 1, \ldots, n \right\}$$

and

$$\bar{Y} = \left\{ y\colon y \epsilon E^n,\ y \geq 0,\ \sum_{i=1}^{n} y_i = Y,\ y_i \text{ an integer for } i = 1, \ldots, n \right\}.$$

The objective for each variant is to find the max-min or the min-max of the appropriate function. As will be seen, a general dynamic programming approach will be applicable to all these models and therefore the computational rules for one model will apply to the rest. The variant that will be used to illustrate the approach is the following

$$\max_{y \epsilon \bar{Y}} \min_{x \epsilon \bar{X}} g(x, y).$$

OPTIMAL *X*-STRATEGY

For any given y-strategy it is possible to obtain an optimal x-strategy for the above variant by using dynamic programming. Suppose we let $\phi_k(\xi, y)$ represent the payoff or loss associated with the optimal allocation of ξ units to any collection of k components of x; that is, for the set of k targets specified by the permutation of the subscripts $\{i_j : j=1, \ldots, k\}$, and for any integer $\xi \epsilon [0, X]$, define

$$\phi_k(\xi, y) = \min_{\sum_{j=1}^{k} x_{i_j} = \xi} \sum_{j=1}^{k} g_{i_j}(x_{i_j}, y_{i_j}), \quad k=1, 2, \ldots, n.$$

This can be written in the standard recursive relation of the functional equation of dynamic programming as

$$\phi_k(\xi, y) = \min_{0 \le x_{i_k} \le \xi} (g_{i_k}, y_{i_k}) + \phi_{k-1}(\xi - x_{i_k}, y)), \quad k=1, \ldots, n.$$

Thus, the x-strategy optimization for any given y-strategy is easily obtained by finding $\phi_n(X, y)$.

The objective of the max-min problem is to determine a y-strategy satisfying

$$\max_{y \epsilon \bar{Y}} \phi_n(X, y).$$

One way of solving this problem is by computing the $\phi_n(X, y)$ for each y-strategy and then selecting the strategy which yields $\max_{y \epsilon \bar{Y}} \phi_n(X, y)$. Since there are $\binom{n+Y-1}{Y}$ such y-strategies, a complete enumeration, though possible, is totally unfeasible since this number quickly becomes quite large. Thus, a modification of the dynamic programming process is proposed to reduce the required number of computations.

THE MAX-MIN ALGORITHM

Suppose we let

$$c_1(\xi, \eta) = \max_{0 \le y_{i_1} \le \eta} \min_{0 \le x_{i_1} \le \xi} g_{i_1}(x_{i_1}, y_{i_1})$$

for integer values of $\xi \epsilon [0, X]$, $\eta \epsilon [0, Y]$, x_{i_1}, and y_{i_1}, where i_1 denotes an arbitrarily chosen component of the vectors. After choosing a second arbitrary component, denoted by i_2, $i_2 \neq i_1$, we can define

$$c_2(\xi, \eta) = \max_{0 \le y_{i_2} \le \eta} \min_{0 \le x_{i_2} \le \xi} (g_{i_2}(x_{i_2}, y_{i_2}) + c_1(\xi - x_{i_2}, \eta - y_{i_2}))$$

as the recursion relation for the first two components. This leads to the general dynamic programming type of recursion relation for any collection of k components of the integer vectors x and y as

$$c_k(\xi, \eta) = \max_{0 \le y_{i_k} \le \eta} \min_{0 \le x_{i_k} \le \xi} (g_{i_k}(x_{i_k}, y_{i_k}) + c_{k-1}(\xi - x_{i_k}, \eta - y_{i_k})).$$

The objective is to find $c_n(X, Y)$.

One might be tempted to interpret $c_k(\xi, \eta)$ in the following way: no matter what the policy might be for the last $n-k$ components of the vectors, the policy over the first k components will be max-

min. This, however, may not be true for $k \geq 3$. For two components $c_2(\xi, \eta)$ does yield the max-min value, but for $k \geq 3$, $c_k(\xi, \eta)$ need not give the true max-min value, and, in fact, it is possible to have different values of $c_k(\xi, \eta)$ for different permutations of the k components.

The difficulty, of course, occurs because insufficient information is stored in the backwards induction process of dynamic programming. By optimizing each time an x or y component is introduced rather than waiting with the optimization process until all vectors can be examined, it is possible for an upward bias to be introduced in the max-min payoff and this bias cannot be corrected in later calculations; that is,

$$c_k(\xi, \eta) \geq \max_{y \in \bar{Y}^k} (\phi_k(\xi, y)),$$

where

$$\bar{Y}^k = \left\{ y : y \epsilon E^k,\ y \geq 0,\ \sum_{j=1}^{k} y_{i_j} = \eta,\ y_{i_j} \text{ an integer for } j = 1, 2, \ldots, k \right\}.$$

Therefore, any value of $c_n(X, Y)$, as calculated by the modified dynamic programming procedure, forms an upper bound to the max-min problem; in other words, we have

THEOREM 1: $c_n(X, Y) \geq \max_{y \epsilon \bar{Y}} \phi_n(X, y)$.

Suppose we let (x^0, y^0) be the integer vectors giving $c_n(X, Y)$; that is, we define (x^0, y^0) by

$$g(x^0, y^0) = c_n(X, Y).$$

If we fix the y^0 vector, then, by the standard x-strategy dynamic programming minimization procedure, we can find

$$\phi_n^0(X, y^0) = \min_{x \epsilon \bar{X}} \sum_{i=1}^{n} g_i(x_i, y_i^0).$$

Now, if (x^0, y^0) is the max-min strategy, then $\phi_n^0(X, y^0) = g(x^0, y^0)$. If y^0 is not the max-min y-strategy, then the y-player is not acting optimally and there exists an x-strategy that will reduce the payoff value below the max-min payoff. Thus, we can state Theorem 2.

THEOREM 2: $\phi_n^0(X, y^0) \leq \max_{y \epsilon \bar{Y}} \phi_n(X, y)$.

In other words, the x-strategy minimization against the y^0 vector yields a lower bound to the max-min payoff value.

The impact of Theorems 1 and 2 is obvious. By means of the not-quite-legal dynamic programming operation we can obtain $c_n(X, Y)$, which gives an upper bound to the desired solution. Then using the vector y^0 which yields this value of $c_n(X, Y)$, we can find the optimal x-strategy against the y^0 vector by a legal application of dynamic programming. This gives a lower bound, $\phi_n^0(X, y^0)$, to the max-min damage value. Then if $c_n(X, Y) = \phi_n^0(X, y^0)$, we can use (x^0, y^0) as the max-min strategy. If $c_n(X, Y) \neq \phi_n^0(X, y^0)$, then the procedure can be repeated using a different random permutation of the vector components in the max-min dynamic programming process. Generally, this will introduce a different upward bias quantity and a new solution (x^0, y^0). Using the new y^0 strategy, a new $\phi_n^0(X, y^0)$ can be found. Then, if $\min c_n(X, Y) - \max \phi_n^0(X, y^0)$ is zero, the strategy $(\hat{x}, y^0)$ defined by

$$g(\hat{x}, y^0) = \max \phi_n^0(X, y^0)$$

is the max-min solution; however, if $\min c_n(X, Y) - \max \phi_n^0(X, y^0)$ is not zero then it is still possible for this difference to be sufficiently small so that the strategy $(\hat{x}, y^0)$ can be used as the max-min solution. The details for choosing when this difference is sufficiently small is a topic of optimal stopping rules. Since stopping rules have been presented elsewhere [4, 5], they need not be repeated here.

EXAMPLE. Danskin in [2] and [3] poses the attack-defense problem of finding the x- and y-strategy yielding

$$\max_{x\in\bar{X}} \min_{y\in\bar{Y}} \sum_{i=1}^{n} f_i(x_i, y_i),$$

where

$$f_i(x_i, y_i) = v_i(1 - \alpha_i e^{-k_i x_i/y_i}) y_i,$$

$$\bar{X} = \left\{ x : x\epsilon E^n,\ x \geq 0,\ \sum_{i=1}^{n} x_i = X \right\},$$

and

$$\bar{Y} = \left\{ y : y\epsilon E^n,\ y > 0,\ \sum_{i=1}^{n} y_i = Y \right\}.$$

If we let $k_i = \log q$ for all i, then q can be interpreted as the probability that a defending unit will be unable to intercept an attacking unit. If we assume that x_i and y_i are integers, then there will be an allocation of $[x_i/y_i] = a_i$ interceptors against each of $y_i(a_i + 1) - x_i = m_i$ attackers and an allocation ot $a_i + 1$ interceptors against the remaining $y_i - m_i$ attackers, where $[\ ]$ denotes the usual greatest integer function. Then the integer form of Danskin's problem is to find

$$\max_{x\in\bar{X}} \min_{y\in\bar{Y}} \sum_{i=1}^{n} f_i(x_i, y_i),$$

where

$$f_i(x_i, y_i) = \begin{cases} v_i(1 - \alpha q^{a_i})^{m_i}(1 - \alpha q^{a_i+1})^{y_i - m_i} & y_i > 0 \\ v_i & y_i = 0 \end{cases}$$

$$\bar{X} = \left\{ x : x\epsilon E^n,\ x \geq 0,\ \sum_{i=1}^{n} x_i = X,\ x_i \text{ an integer for } i = 1, 2, \ldots, n \right\},$$

and

$$\bar{Y} = \left\{ y : y\epsilon E^n,\ y \geq 0,\ \sum_{i=1}^{n} y_i = Y,\ y_i \text{ an integer for } i = 1, 2, \ldots, n \right\}.$$

The max-min variant, which is applicable to the ballistic missile attack-defense problem mentioned earlier, is to find

$$\max_{y\in\bar{Y}} \min_{x\in\bar{X}} \sum_{i=1}^{n} g_i(x_i, y_i),$$

where

$$g_i(x_i, y_i) = v_i - f_i(x_i, y_i)$$

and $\bar{X}$ and $\bar{Y}$ are defined as above. The $g_i(x_i, y_i)$ function is the expected damage value or loss at the i-th target if the target is attacked by y_i offense units and defended by x_i defense units.

Consider the following numerical example of the ballistic missile problem. Suppose three targets with values of 100, 80, and 50 are defended by 40 defenders against an attack of 26 attackers, where the probability of a defender successfully intercepting an attacker is 0.8 and the attacker, if not intercepted, is certain to destroy the target. This implies $v_i = 100$, $v_2 = 80$, $v_3 = 50$, $X = 40$, $Y = 26$, $q = 0.2$, *and* $\alpha = 1$. Because of the commutativeity of addition in the combining of the first two targets using the discrete max-min algorithm, there are only three possible permutations of combining the target components that will give different losses in the modified dynamic programming calculations: (1, 2, 3), (2, 3, 1), and (1, 3, 2). With these data the following results are obtained:

Permutation	$x^0 = (x_1^0, x_2^0, x_3^0)$	$y^0 = (y_1^0, y_2^0, y_3^0)$	$c_n(X, Y) = g(x^0, y^0)$	$\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$	$\phi_n^0(X, y^0) = g(\hat{x}, y^0)$
(1, 2, 3)	(26, 14, 0)	(13, 7, 6)	111.064	(0, 22, 18)	106.239
(2, 3, 1)	(25, 0, 15)	(11, 9, 6)	116.395	(22, 18, 0)	110.773
(1, 3, 2)	(40, 0, 0)	(17, 9, 0)	119.179	(0, 40, 0)	100.739

Since min $c_n(X, Y) = 111.064$ and max $\phi_n^0(X, y^0) = 110.773$, the max-min loss is in the interval [110.773, 111.064]. The discrete max-min algorithm specifies that $y^0 = (11, 9, 6)$ and $\hat{x} = (22, 18, 0)$ should be chosen as the attack-defense strategy with the resulting damage value of 110.773. Actually, a complete enumeration of all y-strategies reveals that the max-min loss is 110.773, and the max-min strategy is the strategy indicated by the algorithm. Note, however, that even if a complete enumeration had not been carried out, we would still be certain that the strategy selected by the algorithm yields a payoff that differs by no more than $111.064 - 110.773 = 0.291$ from the true max-min loss value.

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STOCHASTIC PROGRAMMING MODELS FOR SCHEDULING AIRLIFT OPERATIONS*

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ABSTRACT

This article describes an analytic approach to flight scheduling within an airlift system. The model takes explicit account of the uncertainty present in cargo requirements or demands. For computational feasibility, the approach consists of two related models: (1) a monthly planning model that produces an initial schedule, and (2) a daily model for making periodic changes in the schedule. Both are formulated as two-stage stochastic linear programs. A detailed mathematical description of each model and its physical interpretation is given.

The monthly model determines the number of flights each type of aircraft in the fleet. Excess demands on certain routes are assumed to be met, at least in part, by spot procurement of commerical lift from outside the system. The flight assignment is determined by minimizing the expected total system cost, which consists of operating costs, costs of reallocating aircraft to different routes, spot commercial procurement costs, and other penalty costs of excess demand. The model accounts for limitations on the number of flying hours and the carrying capacities of various aircraft in satisfying demands.

In the daily model the number of aircraft of each type to switch from one route to another and the number of commercial flights on spot contract to add on the current day are the principal decision variables. These are determined by balancing operating, procurement, and redistribution costs against the expected costs of additional cargo delay. The current state of the system—the amount of unmoved cargo on various routes andathe position of aircraft throughout the system—plays a role in determining these decisions.

A description of two variants of an algorithm recently developed for this class of problems is presented. Both versions, which use ideas from convex programming, make extensive use of linear programming codes for the brunt of the calculations. The models may thus be solved by augmenting existing linear programming routines.

I. INTRODUCTION

This paper provides the technical details of a mathematical model for scheduling flights in a large airlift system. The model takes explicit account of the uncertainty surrounding the requirements at the time scheduling decisions must be made, as well as of the various physical constraints on the system. The proposed models are adaptive in the sense that current information about the state of the system (cargo levels at various origins, distribution of vehicles throughout the system) is used periodically in making decisions regarding changes in the flight schedule.

The approach discussed here consists of two components—a monthly and a daily flight planning model. Both are formulated as stochastic two-stage linear programs, and can be solved by algorithms based upon standard linear programming calculations. Consequently, the model can be applied to large-scale systems. This should permit scheduling the system as a whole rather than the two divisions separately, with transfers of aircraft between then on an ad hoc basis, as is currently the practice.

*This research is supported by the United States Air Force under Project RAND. Views or conclusions contained in this study should not be interpreted as representing the official opinion or policy of the United States Air Force.

The paper is organized as follows. The next section contains a brief description of the scheduling problem and discusses the overall strategy employed in treating it analytically. This is followed in Sec. III by a mathematical statement of the monthly and daily models, which are similar in general structure. Section IV contains a discussion of two variants of an algorithm for solving these models, preceded by a short discussion of pertinent concepts for two-stage linear programming under uncertainty. In Sec. V we present a small numerical example for the monthly model.

II. PROBLEM STATEMENT

Description of the Airlift System

Airlift missions within the system are of two basic types: channel missions and special missions. Channel missions fly on a regular basis between fixed origins and destinations: Fixed airports in the continental United States, aerial ports of embarkation (APOEs) and overseas bases, and aerial ports of debarkation (APODs). They carry cargo and passengers.* They are scheduled in advance for each calendar month, although the schedules (number of flights and departure times) are changed during the month, often daily, based upon deviations from expected cargo generation. Many of these flights make intermediate stops for refueling and crew changes. Channel cargo is shipped into APOEs for storage until shipped.

Special missions are flights from points of original cargo generation, such as factories or warehouses, to points at or near the ultimate consignee. Such flights arise on an irregular basis, and are scheduled if sufficient advance notice is provided. The time and place of cargo pickup is usually specified. Often these flights arise after the beginning of the month and require a temporary diverting of aircraft from channel missions, since certain special missions are designated higher priority than channel traffic. The users of a special mission must, in effect, charter an entire aircraft, although the airlift operator decides what type of aircraft will be supplied.

The system is divided administratively into two operating divisions. One division comprises APOEs on the West Coast and serves points in Asia. The other comprises APOEs on the East Coast and serves bases in Europe, the Middle East, and South America, although currently several channels serve destinations in Asia. Headquarters assigns special missions to the two divisions, usually on the basis of whether the cargo originates west or east of the Mississippi River. Often aircraft belonging to one division are temporarily assigned to the other division to meet these demands when the type of aircraft required is in short supply.

Scheduling Problem

In carrying out its operations, the operator is faced with random fluctuations in the amount of cargo to be carried over its various routes.

There is monthly uncertainty because shippers' forecasts of requirements for the month as reported to the airlift operator at various lead times differ significantly from the actual amount of requirements shipped. Secondly, the amount of cargo actually received during a day is partly unpredictable due to random variability in production schedules for manufactured items, delays in transit and depot processing, and other factors.

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*Due to the large size of the system only cargo flights are considered in the models to be developed. (Also most of the passengers are carried by commercial flights.) Passengers can be handled in the models developed below by converting them to ton (or cube) equivalents if flights can carry mixed loads. Otherwise demand would have to be disaggregated into separate commodity classes significantly increasing the size of the models.

A scheduling problem arises, since demand is not known with certainty and aircraft differ with respect to their payloads, flying times, and operating costs over different routes. Therefore, aircraft of different types must be assigned to various routes over the monthly planning period. In addition, the system can obtain additional lift from commercial sources on call contracts to handle excessive demands on some routes. Furthermore, the assignments of aircraft cannot violate constraints for flying hours.

As time unfolds and demand becomes known, the scheduler can transfer those aircraft from routes where demand is less than anticipated to those where it is higher than expected. Thus, the scheduling problem is a dynamic one of periodically reassigning capacity to routes in order to minimize the expected costs of the operation over time. Assigning insufficient airlift increases both the costs due to delay and the procurement costs of "emergency" commercial lift; assigning excess capacity results in an opportunity cost of foregoing service elsewhere in the system. Since both situations are stochastic, the best that can be done is to minimize the cost entailed in these situations on the average.

Basic Approach

We turn now to an overview of our strategy in treating the scheduling problem. Ideally, the problem should be treated as a multiple period (30 days or longer), sequential decision problem with account being taken of daily demand variation and physical constraints on aircraft in each of the ensuing periods in the current horizon. Such an approach is not feasible from a computational standpoint for a system the size of the one considered or even each division separately. Therefore, broadly following the current procedures, we break the process into two parts—monthly and daily.

A monthly schedule is determined that accounts for uncertainty for the forthcoming month considered as one aggregate period in which decisions planned in advance may be modified. Although this is far from ideal, it is superior to ignoring uncertainty and the subsequent opportunity to remedy its effects. The second-state decisions, as we shall see, are "virtual" or fictitious, but are nonetheless important because they help provide a good first-stage decision, the initial assignments that serve as the starting point for the daily modifications.

The monthly flights are apportioned to the days. For example, if 15 flights of a given aircraft are found from the monthly model, this may be interpreted as one flight per aircraft every two days during the month if the given aircraft type can fly the mission.

Once the month commences, daily amendments to the number of flights scheduled for that particular day are determined by the daily model, which is also a two-stage optimization. Changes in assignments optimal for the forthcoming 2-day period are determined. They minimize the sum of costs incurred on the current day, given the amount of cargo on hand plus the expected value of tomorrow's costs, which depend upon both the amount of new cargo and the changes made on the first day. Consecutive decisions of the daily model are thereby chained together in a recursive fashion with outputs from the model for day t becoming inputs to the model for day $t+1$.

III. MATHEMATICAL FORMULATION

In this section we formulate the monthly and daily models as two-stage linear programs under uncertainty [2], [10] with a random right-hand side in the second stage. Several of the program's variables represent numbers of different kinds of flights and consequently must be integer valued. However, this aspect of the problem is neglected in order to obtain a program that is computationally solvable, thus leaving the user to adopt a rounding scheme. The general form for problems of this type is the following:

$$\min_{x \geq 0} \left\{ cx + E_\xi [\min_{y \geq 0} qy | Wy = \xi - Tx] \right\} \quad \text{subject to } Ax = b. \tag{1}$$

In this problem, c, q, and b are known vectors of dimension n_1, n_2, and m_1, respectively. A, T, and W are known matrices of dimension $m_1 \times n_1$, $m_2 \times n_1$, and $m_2 \times n_2$, respectively. The vectors x and y represent the first- and second-stage decision vectors. ξ is a random vector with a known probability distribution and $E(\cdot)$ denotes the expectation with respect to that distribution.

The problem can be given the following interpretation. Before the random variable ξ is observed, one must choose nonnegative values of x satisfying $Ax = b$. The immediate cost of this choice is cx. After x is chosen and the random variable ξ is observed, nonnegative values must be chosen for y which satisfy $Wy = \xi - Tx$ at a cost of qy. The chosen value of y will, for each observed value of ξ, be the one which minimizes qy. Letting $Q(x) = E_\xi\{\min_{y \geq 0} qy | Wy = \xi - Tx\}$, we wish to minimize the first-stage cost, cx, plus the expected second-stage cost, $Q(x)$. The problem can be viewed as a dynamic programming problem with only two stages, with y a function of the state vector $\xi - Tx$. Note that x does not depend on the observed values of ξ but only on the distribution of ξ, while y does depend on the observed values of ξ and on the chosen values of x.

Monthly Model

In the monthly model the sequence of decisions is the following:

1. The number of flights of each aircraft type in the MAC fleet over each route* is assigned before monthly requirements** on each route are known with certainty.

2. After monthly requirements are observed, some flights assigned to routes with lower than expected demand are switched to routes with higher than expected demand. Commercial airlift is also sometimes added to routes with excess demand.

The monthly model has the following structure. The first-stage constraints state that for each aircraft type the total number of flying hours allocated to all routes cannot exceed the total number of flying hours available of that type. The second-stage constraints are of two types. The first specifies that the number of flying hours of a given aircraft type diverted from a particular route to other routes cannot exceed those initially assigned to it. The second type are demand balance equations, which state that for each route the total carrying capacity (that originally assigned, minus that diverted to other routes plus that diverted from other routes) minus unused carrying capacity plus unsatisfied demand is equal to total demand for that route.

The objective function consists of the cost of the final flying program (the initial plus the amended assignments) plus penalty costs of excess demand or supply. The cost of excess demand is reflected in both the cost of additional commercial lift plus the extra flying time consumed in switching aircraft from one route to another. Specifically, the program is as follows.

$$\text{Find min } Z,\ x_{ij},\ x_{ijk},\ y_j^+,\ y_j^- \geq 0 \qquad \text{such that}$$

*A route is a channel or special mission from each origin. Routes are assumed to be point-to-point with no intermediate on-load or off-load of cargo; however, e.g., a trip from San Francisco to Japan via Hawaii is considered a different route from a flight which goes via Alaska.

**Requirements as used here will mean net requirements from which capacity of commercial airlift on an annual contract (for which the frequency of trips and routes is fixed) and any other fixed capability that is available is subtracted. This has the effect of merely lowering the mean of the distribution of requirements.

$$(2)\qquad \sum_{i,j} c_{ij}x_{ij} + E\left\{ \min\left[\sum_{i,j,k\neq j} \left(c_{ijk} - c_{ij}\frac{a_{ijk}}{a_{ij}} \right) x_{ijk} + \sum_j c_j^+ y_j^+ + \sum_j c_j^- y_j^- \right] \right\} = Z.$$

First stage:

$$(2.1)\qquad \sum_j a_{ij}x_{ij} \leq F_i, \qquad \text{all } i.$$

Second stage:

$$(2.2)\qquad x_{ij} - \sum_{k\neq j} \frac{a_{ijk}}{a_{ij}} x_{ijk} \geq 0, \qquad \text{all } i, j$$

$$(2.3)\qquad \sum b_{ij}x_{ij} - \sum_{i,k\neq j} \left[b_{ij}\left(\frac{a_{ijk}}{a_{ij}} \right) \right] x_{ijk} + \sum_{i,k\neq j} b_{ij}x_{ijk} - y_j^+ - y_j^- = d_j, \qquad \text{all } j,$$

where x_{ij} = number of flights of aircraft type i initially assigned to route j during the month;

x_{ijk} = number of flights of aircraft type i assigned to route k using hours made available by canceling route j flights;

y_j^+ = demand on route j which is satisfied by commercial lift, if permitted, or unsatisfied demand if commercial lift not permitted;

y_j^- = unused capacity on route j;

and

a_{ij} = number of hours required by aircraft type i for a flight initially assigned to and flown on route j;

a_{ijk} = number of hours required by aircraft type i for a flight on route k that uses hours made available by canceling route j flights ($a_{ijk} \geq a_{ik}$);

b_{ij} = the carrying capacity (tons or any other appropriate measure) of a flight of an aircraft of type i on route j;

F_i = the maximum number of flying hours for aircraft of type i available during the month;

d_j = total demand (tons or any other measure) for route j (a random variable);

c_{ij} = cost of flight of aircraft type i initially assigned to and flown on route j ($c_{ij} \geq 0$);

c_{ijk} = cost per flight of aircraft type i assigned to route k from hours made available by canceling route j flights ($c_{ijk} \geq c_{ik}$);

c_j^+ = cost per ton of commercial augmentation on route j. If commercial augmentation is not available to carry excess demand, this may instead represent a shortage cost;

c_j^- = cost of a unit of unused carrying capacity on route j.*

Note that since a flight assigned to and flown on route j takes a_{ij} hours while a flight flown on route k from hours diverted from route j takes a_{ijk} hours, such a flight on route k results in the cancellation of (a_{ijk}/a_{ij}) flights on route j. Thus, of the flights initially assigned to route j, those that are actually flown number

$$x_{ij} - \sum_{k\neq j} \frac{a_{ijk}}{a_{ij}} x_{ijk}.$$

This is reflected in expressions (2), (2.2), and (2.3).

---

*This model and the daily model described below represent generalizations of the pioneering Ferguson-Dantzig model [4] applying linear programming under uncertainty to aircraft allocation. In contrast to their model, the monthly model considers reassignment of aircraft once demand is known and hence its second stage consists of a bonafide linear program. Also the demand distributions in our models are continuous rather than discrete.

The first-stage constraints correspond to $Ax=b$ and the second-stage constraints to $Wy=\xi-Tx$ in expression (1). Notice that the constraints (2.2) are placed in the second stage despite the fact that the right-hand side is not a random vector. This is done because the switching variables x_{ijk} are assumed to be determined *after* demand becomes known and thus they depend upon information about the d_j. Also, they are to be chosen simultaneously with the y_j^+ and y_j^- to minimize the overall second-stage cost function.

Daily Scheduling Model

The daily scheduling model is also a two-stage linear program under uncertainty. Its principal first-stage decision variables are the number of flights of each type to switch from one route to another in the forthcoming day—variables that played the role of second-stage recourse variables in the monthly model. In addition, the daily model is couched in terms of unmoved cargo stocked at the various origins (aerial ports of embarkation) for various lengths of time; the cargo has not yet been shipped, and subsequently incurs certain delay costs. Consequently, the amount of cargo in storage for various lengths of time waiting to be loaded on different aircraft types over the different routes constitutes another set of first-stage decision variables.

In the first stage the constraints consist of balance limitations on the number of flying hours, bounds on switches, capacity restrictions on shipments, and inventory balance equations for cargo which has been in the port for over 1 day—quantities known with certainty. The second stage consists of demand equations for newly arriving cargo which is treated as a random variable. These equations include terms to account for a possible under- or over-supply of capacity. It will be assumed that for a given route new cargo has a lower priority (i.e., lower penalty or delay cost) than old cargo. The reason for this will become clear later.

The objective function is again one of cost minimization. In this case the costs include (1) incremental operating costs resulting from flight changes, (2) daily commercial procurement, (3) delay costs for cargo that remains unmoved for various lengths of time, and (4) expected penalty costs resulting from misallocation of capacity.

To write the daily model, we introduce the following notation (variables refer to the t-th day of the month):

x_{ij} = number of flights of aircraft type i previously scheduled to fly over route j, a known constant;

x_{ijk} = number of flights of aircraft type i switched to route k using hours made available by canceling route j flights;

y_{ij}^l = capacity reserved for cargo l days old shipped on aircraft type i over route j ($l=0, 1, \ldots, L$);

w_{hj}^l = capacity reserved for cargo l days old shipped on commercial lift of type h over route j;

z_{hj} = number of flights of commercial lift of type h procured for route j;

s_j^l = amount of cargo l days old remaining to be shipped over route j at the end of the day;

H_j^l = inventory of cargo l days old for route j for shipment at the beginning of the day (H_j° represents new cargo and is a random variable);

v_j = unused capacity on route j at the end of the day;

a_{ij} = number of hours required by aircraft type i scheduled and flown on route j;

a_{ijk} = number of hours required by aircraft type i for a flight on route k using hours made available by canceling route j flights ($a_{ijk} \geq a_{ik}$);

b_{ij} = the carrying capacity of aircraft type i over route j;

c_{ij} = cost per flight of aircraft type i scheduled and flown on route j ($c_{ij} \geq 0$);

c_{ijk} = cost per flight of aircraft type i assigned to route k using hours made available by canceling route j flights ($c_{ijk} \geq c_{ij}$);

β_{hj} = cost per flight of commercial lift of type h over route j;

γ_j^l = unit cost per ton of one additional day's delay for cargo l days old which is to be shipped over route j. It is assumed that $\gamma_j^\circ = \min_l \gamma_j^l$.

Then the problem for day t can be stated. Find

$$\text{Min } Z,\ x_{ijk},\ s_j^l,\ z_{hj},\ y_{ij}^l,\ w_{hj}^l,\ v_j \geq 0,$$

such that

$$\sum_{i,j,k\neq j}\left(c_{ijk}-c_{ij}\left(\frac{a_{ijk}}{a_{ij}}\right)\right)x_{ijk}+\sum_{j,l\neq 0}\gamma_j^l s_j^l+\sum_{h,j}\beta_{hj}z_{hj}+E\left\{\min_{s_j^\circ}\sum_j \gamma_j^\circ s_j^\circ\right\}=Z. \tag{3}$$

First Stage:

$$\sum_{k\neq j}\frac{a_{ijk}}{a_{ij}}x_{ijk}\leq x_{ij},\ \text{all } i \text{ and } j; \tag{3.1}$$

$$\sum_{l>0}y_{ij}^l+y_{ij}^\circ+\sum_{k\neq j}\left(b_{ij}\frac{a_{ijk}}{a_{ij}}\right)x_{ijk}-\sum_{k\neq j}b_{ij}x_{ikj}\leq b_{ij}x_{ij},\ \text{all } i \text{ and } j; \tag{3.2}$$

$$\sum_{l>0}w_{hj}^l+w_{hj}^\circ-b_{hj}z_{hj}\leq 0,\ \text{all } h \text{ and } j; \tag{3.3}$$

$$\sum_i y_{ij}^l+\sum_h w_{hj}^l+s_j^l=H_j^l,\ \text{all } j \text{ and } l>0. \tag{3.4}$$

Second Stage:

$$s_j^\circ-v_j=H_j^\circ-\sum_i y_{ij}^\circ-\sum_h w_{hj}^\circ,\ \text{all } j. \tag{3.5}$$

Equation (3.1) states, as in the monthly model, that the number of flying hours switched off a given route and aircraft type cannot exceed those originally assigned.

The meaning of Eqs. (3.2) and (3.3) is that the total tonnage loaded of cargo already received plus the amount reserved for new cargo on a given aircraft type over a given route must be less than the carrying capacity of all aircraft of that type assigned to the route and for the commercial vehicles, respectively.* Equation (3.4) states for each duration of cargo destined for each route, the total tonnage allocated for the aircraft and commercial airlift plus the amount not moved must equal the inventory of cargo at the beginning of the day (a known constant for all but newly arriving cargo). We note that the assumption that newly arrived cargo has lowest penalty cost (mathematically stated, $\gamma_j^\circ = \min_l (\gamma_j^l)$ for all j) assures that no new cargo will be carried unless all old cargo is carried. This assures that for all but newly arrived cargo, the amount shipped uses all the allotted capacity. If this assumption were not made, it would be more reasonable to determine the y_{ij}° and w_{ij}° simultaneously with the y_{ij}^l and $w_{ij}^l (l>0)$ as a function of the actual amount of new cargo. This would necessitate shifting equations (3.2) through (3.4) into the second stage, making for a large linear program for the second stage. The present formulation permits the second stage to have a very simple structure which reduces the computational burden in solving it. This matter will be discussed in more detail below.

*For routes on which commercial augmentation is not permitted, the corresponding z_{hj} are omitted.

Finally, Eq. (3.5) states that for each route the inflow of new cargo minus the total capacity reserved for shipment of new cargo on all types of operator-owned and commercial vehicles is equal to any cargo not shipped minus any unused capacity.

In addition, we may wish to include constraints that take into consideration the capacity limitations of enroute bases. Let $J(p)$ equal the set of routes which use base p as an enroute stop for refueling or crew change. Let α_p be the maximum number of flights that can be accommodated at base p in a single day. Then these constraints can be expressed as

$$\sum_{i,\, j \epsilon J(p)} \left(x_{ij} - \sum_{k \neq j} \frac{a_{ijk}}{a_{ij}} x_{ijk} + \sum_{k \neq j} x_{ikj} \right) \leq \alpha_p.$$

Letting $H_j^l(t)$ be the inventory of cargo for route j that is l days old on day t, and letting quantities in parentheses show time dependence of other quantities in a similar way, note that $H_j^l(t+1) = s_j^{l-1}(t)$, using the convention that $s^{l-1}(0)$ refers to cargo not shipped on the last day of the preceding month.

In addition, the quantities x_{ij} are updated daily by the relations

$$x_{ij}(t+1) = x_{ij}(t) - \sum_{k \neq j} \frac{a_{ijk}}{a_{ij}} x_{ijk} + \sum_{k \neq j} x_{ikj},$$

where the x_{ijk} and x_{ikj} are the optimal values of the variables as determined by the model for day t.

In the monthly model there are m (number of individual aircraft types) constraints in the first stage and $(mn+n)$ in the second stage (n being the total number of routes). The assumption that there are five aircraft types and 100 routes presents a linear program for the first stage with five constraints and 600 for the second stage; however, in the second stage the first mn are of the generalized upper-bound type, to be discussed later, to which special methods may be applied.

In the daily model there are at most $n(2m+H+L)$ constraints in the first stage and n in the second stage, where H is the number of individual commercial vehicles and L is the number of duration classes of old cargo. For $H=3$, $L=3$, $m=5$, and $n=100$ this yields one first-stage program with 1600 constraints. The second stage contains 100 equations. While this is within the range of linear programming codes for "third generation" computers, the first-stage problem can be simplified by treating it by decomposition methods where Eq. (3.2) is the coupling equation and (3.1), (3.3) and (3.4) are the two disjoint subproblems. Again, for the former subproblem, the majority of the constraints (3.1) are of the generalized upper-bound variety.

IV. SOLUTION TECHNIQUES

This section describes two alternative computational methods for solving two-stage linear programs under uncertainty. These methods may be applied to the monthly and daily model. They assume the random variables have a continuous distribution. Each is a variant of an algorithm for convex programming, which is required for this type of problem. The first is a cutting-plane or constraint-generating procedure, while the second is a gradient method. We outline the steps required for the general problem exemplified by the monthly model and indicate certain simplifications that can be made in the daily model whose second-stage has a special structure. Before proceeding to the algorithms, we review some salient facts concerning two-stage linear programming under uncertainty.

Basic Facts from Linear Programming Under Uncertainty

As mentioned in Sec. III, both the monthly and daily problems are special cases of the problem:

(4) Find $x \geq 0$, min Z such that

$$Z = cx + Q(x)$$

$$Ax = b$$

$$\text{where } Q(x) = E\left[\min_{y \geq 0} qy \mid Wy = \xi - Tx\right].$$

Wets [9] has shown that $Q(x)$ is convex in x. Thus the above linear program under uncertainty is equivalent to a deterministic convex program in x. The algorithms to be developed focus and are based on certain properties of the function $Q(x)$.

One important property (to be proved later in the Constraint Generation section) is that the linear function $\rho(x^\circ) - (\pi(x^\circ)T)x$ is a support function* (or tangent plane) to $Q(x)$ at $x^\circ \epsilon K$, where K is $\{x \mid Ax = b\}$ and $\pi(x^\circ)$ is the expectation of the optimal dual variables and $\rho(x^\circ)$ is the expectation of the demand terms in the dual objective function. Specifically, let $\pi(x^\circ, \xi)$ be an optimal solution to

(5) $$\max_{\pi} \left\{ \pi(\xi - Tx^\circ) \mid \pi W \leq q \right\}.$$

Then

$$\pi(x^\circ) = E_\xi \pi(x^\circ, \xi) \text{ and } \rho(x^\circ) = E_\xi\{\pi(x^\circ, \xi)\xi\}.$$

This property forms the basis for the cutting-plane method.

Another important property [9] is that if the distribution of ξ is absolutely continuous, $Q(x)$ is differentiable at $x^\circ \epsilon K$ and has the gradient $-\pi(x^\circ)T$, where again $\pi(x^\circ)$ is the expected value of the solution to (5). This property is utilized in the gradient approach developed by Wets [9].

In general, the $\pi(x^\circ)$ and $\rho(x^\circ)$ cannot be explicitly computed; consequently, the methods to be described resort to sampling procedures to estimate these quantities.

Our monthly and daily problems both possess the property of *complete recourse*. In other words, each x satisfying $Ax = b$ yields a feasible program for all possible ξ in the second-stage constraints $Wy = \xi - Tx$. This permits us to eliminate one of the steps in the algorithms to be described for finding feasibility cuts [8, pp. 31–34]. Moreover, the daily model exhibits simple recourse, i.e., $W = [I, -I]$. This permits simplification of one major step in the following algorithm.

Cutting-Plane Method

A cutting-plane algorithm has been developed by Wets and Van Slyke [8] for solving the general problem. It can be viewed as an application of Kelley's cutting-plane method [6] to the equivalent convex program.

In this approach the convex function $Q(x)$ appearing in the objective function is treated as a constraint. The objective function is replaced by $z(x) = cx + \theta$ and the constraint,

(6) $$Q(x) \leq \theta,$$

considered in the first-stage program where θ is treated as an additional decision variable. Since the second-stage costs are nonnegative in our problem, we may consider θ as a nonnegative variable.

*A linear function $H(x)$ is a support for convex function $z(x)$ defined on $X \subset E^n$ at x° if $z(x) \geq H(x)$ for all $x \epsilon X$ and $z(x^\circ) = H(x^\circ)$.

The constraint (6) is represented by a set of support planes $\rho^k - (\pi^k T)x$; i.e., a convex function is represented as the upper envelope of its supporting planes. Four constraints are generated by solving the second-stage problem for a fixed x which solves an augmented first-stage problem. Thus the technique consists of a decomposition of the overall problem into two linear-programming subproblems. The method can be viewed, in fact, as a particular case of Bender's partitioning method [1] applied to the original problem.

The particular problem that is solved at any iteration, say the k^{th}, is the following:

1. Find $x, \theta \geq 0$, min $\tilde{z}$ such that

$$
\begin{aligned}
& cx + \theta = \tilde{z} \\
& \text{a) } Ax = b \\
& \text{b) } (\pi^k T)x + \theta \geq \rho^k, \qquad k = 1, \ldots, S,
\end{aligned} \tag{7}
$$

where S is an index of a subset of previously generated constraints to be described more fully below.

The constraints (7b) are generated sequentially and have the property that for all x satisfying (7a), $(x, Q(x)) = (x, \theta)$ satisfies (7b).

2. Let $(\bar{x}, \bar{\theta})$ be the solution obtained from step 1 and solve for $Q(\bar{x})$. If $Q(\bar{x}) = \bar{\theta}$, $\bar{x}$ solves (4). Otherwise, a constraint of the type (7b) is generated that is violated by $(\bar{x}, \bar{\theta})$.

Constraint Generation—For fixed $x = \bar{x}$ and a given $\xi = \xi$, the second-stage linear program becomes:

$$
\begin{aligned}
& \text{Find } y \geq 0, Q(\bar{x}, \xi) \text{ where} \\
& Q(\bar{x}, \tilde{\xi}) = \min qy \text{ subject to } Wy = \tilde{\xi} - T\bar{x}.
\end{aligned} \tag{8}
$$

Let $\pi(\bar{x}, \tilde{\xi})$ be the simplex multipliers corresponding to an optimal basis in (8). It follows that $\pi(\bar{x}, \tilde{\xi})(\tilde{\xi} - T\bar{x}) = Q(\bar{x}, \tilde{\xi})$ and $\pi(\bar{x}, \tilde{\xi})W \leq q$. Thus, $\pi(\bar{x}, \tilde{\xi})$ is feasible in the following program for all x:

$$
\begin{aligned}
& \text{Find } \pi, \max w \text{ where} \\
& \pi(\tilde{\xi} - Tx) = w \\
& \pi W \leq q.
\end{aligned} \tag{9}
$$

Since (9) is dual to (8) for general x, it follows from duality theory that

$$\pi(\bar{x}, \tilde{\xi})(\tilde{\xi} - Tx) \leq Q(x, \tilde{\xi})$$

for all x. Letting $\pi(\bar{x}) = E_\xi \pi(\bar{x}, \xi)$ and $\rho(\bar{x}) = E_\xi[\pi(\bar{x}, \xi)\xi]$, it follows that for all x, $(\pi(\bar{x})T)x + Q(x) \geq \rho(\bar{x})$. Hence, the following constraint must be satisfied by all $(x, Q(x))$:

$$(\pi(\bar{x})T)x + \theta \geq \rho(\bar{x}). \tag{10}$$

Algorithm—All constraints of the type (7b) are of type (10). That is, each (π^k, ρ^k) is equal to some $(\pi(\bar{x}), \rho(\bar{x}))$. Since for every x satisfying $Ax = b$, $(x, Q(x))$ is feasible for (7), it follows that in (7), min $\tilde{z} \leq$ min z, where min z solves (4). Thus, if $(\bar{x}, \bar{\theta})$ solves (7) and $Q(\bar{x}) = \bar{\theta}$, $c\bar{x} + Q(\bar{x}) \leq \min z$ and $\bar{x}$ solves (4).

On the other hand, if $Q(\bar{x}) \neq \bar{\theta}$, it follows (since $c\bar{x} + Q(\bar{x}) \geq \min z$) that $Q(\bar{x}) > \bar{\theta}$. Also $(\pi(\bar{x})T)\bar{x} + Q(\bar{x}) = \rho(\bar{x})$ by taking expectations in the expression $\pi(\bar{x}, \tilde{\xi})(\tilde{\xi} - Tx) = Q(\bar{x}, \tilde{\xi})$. Thus $(\bar{x}, \bar{\theta})$ violates

the expression (10).

One problem which arises is that it is possible to generate a large number of constratints of type (7b). However, in reference [7] it is shown that all constraints in (7b) that are not tight (i.e., where equality does not hold) in the optimal solution to (7) may be dropped from succeeding iterations. Thus, the number of such constraints never exceeds the number of rows in T. Formally, the algorithm is as follows with S initially zero in step 1.

1. Solve for $x \geq 0$, min $\tilde{z}$ such that

$$cx + \theta = \tilde{z}$$
$$Ax = b$$
$$(\pi^k T)x + \theta - s^k = \rho^k \qquad k = 1, \ldots, S.$$

Let $\bar{x}$ be the optimal x obtained.

2. If $s^k > 0$, delete the constraint in which s^k appears, $k = 1, \ldots, S$ and renumber those remaining.

3. Sample from the distribution of ξ. For each point $\xi^i (i = 1, \ldots, n)$ in the sample, solve for

$$y \geq 0, Q(\bar{x}, \xi) \text{ where}$$

$$Q(\bar{x}, \xi^i) = \min qy \text{ subject to}$$

$$Wy = \xi^i - T\bar{x}.$$

Increase S by 1 and let $\pi^s = n^{-1}\Sigma\pi(\bar{x}, \xi^i)$ and $\rho^s = n^{-1}\Sigma\xi^i\pi(\bar{x}, \xi^i)$, where n is the sample size and $\pi(\bar{x}, \xi^i)$ are the optimal simplex multipliers for the sample point ξ^i.

4. If $Q(\bar{x}) - \theta \leq \delta$, where δ is a predetermined constant or variable, terminate. Otherwise, return to 1.

The quantity δ in step 4 may be chosen by any sensible criteria. For example, it may be a constant or it could be a small fraction times the last step 1 solution (i.e., $z(\bar{x})$).

When the second stage exhibits simple recourse (i.e., the only second-stage variables are slack variables) as is the case in the daily model, step 3 can be considerably simplified. In this case the sampling procedure and the need to solve a linear program can be dispensed and the expected value of the shadow prices calculated explicitly. Specifically, $Wy = \xi^i - T\bar{x}$ may be expressed as $q_i^+ - q_i^- = \xi^i - Tx$, $i = 1, \ldots, m_2$, and from [8]:

$$E\pi_i(x^k) = q_i^+ \int_{\xi_i \geq T_i x^k} dF(\xi_i) - q_i^- \int_{\xi_i < T_i x^k} dF(\xi_i), \qquad i = 1, \ldots, m_2,$$

where $F(\xi_i)$ is the cumulative of the distribution of ξ_i. That is, they are equal to the cost of the i-th excess demand times the probability of its occurrence minus the cost of the excess supply in the i-th equation times the probability of its occurrence.

The Second-Stage Problem—The sampling involved in the calculation of $Q(\bar{x})$ in step 3 of the algorithm involves the solution of many linear programs. The solution time for this step can greatly be reduced by taking advantage of the fact that if a basis is optimal for a program and the right-hand side is changed, that basis will either be optimal or infeasible for the new right-hand side. Thus, the sequence of programs for the second stage may be solved as follows. Solve the program for one sample point. Check each of the other sample points for feasibility with respect to the optimal basis found.

(The π vector associated with this basis is an optimal one for all sample points that are feasible.) Then solve the program for some sample point that was infeasible and repeat. This procedure reduces the number of linear programs from the number of elements in the sample to the number of optimal bases for the sample points. After one optimal basis is found, others may of course be found from this one by a sequence of dual simplex pivot steps. Hence, each pivot will yield a basis whose cost row is nonnegative. It is suggested, then, that after each iteration, feasibility be checked for the other sample points which have not been solved for, rather than wait until the current one being considered is feasible. It is also suggested that the optimal basis be saved from iteration to iteration.

Gradient Method

In this approach the equivalent convex program is solved by a nonlinear programming algorithm which utilizes the gradient of the objective function of (4) at x°,

$$\nabla z(x^\circ) = (c - \bar{\pi}(x^\circ)T).$$

The method we describe is based upon a variant of the Frank-Wolfe algorithm [5] for convex programming with linear constraints developed by Wets [9] for stochastic programs. Other gradient methods for convex programming, such as the Gradient Projection method, Zangwill's convex simplex method [11], or Zoutendijk's method of feasible directions [12] could be employed as well.

In brief, this method obtains solutions to a sequence of linear approximations to the given problem of the form

$$\min_x \nabla z(x^{l-1})x \tag{11}$$

subject to

$$Ax = b$$
$$x \geq 0,$$

where x^{l-1} is the solution for x to the given problem at the preceding or $(l-1)$st iteration. An improved feasible solution x^l is obtained by adding to x^{l-1} an improving feasible direction $\lambda(\bar{x}^l - x^{l-1})$, where $\bar{x}^l$ is the (basic) optimal solution to (11) and λ(a scalar) is determined by solving a one-dimensional optimization problem to be described below.

Several features of this method should be noted. First, the linear program which is actually solved in the first stage is not augmented but contains only constraints of the first stage; only the cost function varies from iteration to iteration. Second, the computational labor is again split between first and second stages with the expected values of the shadow prices, which are used in the objective function of the first-stage problem, generated in a sampling procedure in solving a set of second-stage programs for a fixed x. Third, since the method is a gradient approach, it is self-correcting in the sense that if in an earlier iteration a poor approximation to the gradient is generated by an unusually large sampling error which leads to a small improvement in the solution, it is likely to be corrected at later iterations. By contrast, in the cutting-plane approach, if an unusually large sampling error generates a support function which cuts inside the feasible region, there is no assurance that future iterations with smaller sampling errors will eliminate this constraint.

Algorithm—The exact steps of the algorithm are as follows:

1. (Initialization.) Set iteration count $l=0$, $x^l=0$ and $\bar{\pi}^{l-0} \equiv \bar{\pi}(x^l)=0$ [the sample estimate of the expected value of the optimal multipliers for the second-stage program].

2. (Generate optimal solution to the linearized equivalent convex program.) Set $l = l + 1$. For $l = 1, 2, \ldots$, solve the linear program

(i)
$$\min_x (c - \bar{\pi}^{l-1}T)x,$$

subject to

$$Ax = b$$
$$x \geq 0.$$

Denote the optimal solution to (i) by $\bar{x}^l$. For $l = 1$, go to step 5. Otherwise go to step 3.

3. (Seek improved direction.) Solve the scalar optimization problem:

(ii)
$$\min_{0 \leq \lambda \leq 1} \psi(\lambda) = c[(1-\lambda)x^{l-1} + \lambda\bar{x}^l]$$
$$+ \bar{Q}[(1-\lambda)x^{l-1} + \lambda\bar{x}^l],$$

where $\bar{Q}$ is a sample estimate of $Q(x)$ obtained by solving the second-stage program for a random sample of ξ and a fixed x. In (ii) this function is viewed as a function of the scalar parameter λ, which weights the previous solution with the current optimal solution to the linearized problem. Denote the optimal solution to (ii) by λ^*.

$$\text{Set } x^l = (1-\lambda^*)x^{l-1} + \lambda^*\bar{x}^l.$$

4. (Test for convergence.) If $\lambda^* = 0$, terminate with x^{l-1} as the optimal solution. Otherwise, set $x^l = (1-\lambda^*)x^{l-1} + \lambda^*\bar{x}^l$ and go to step 5.

5. (Determine $\bar{\pi}^l$.) This is accomplished as in step 3 of the cutting-plane algorithm by sampling from the distribution of ξ and for each solving the resulting linear programs parametrically in the general case and taking the average of the resulting simplex multipliers, or explicitly calculating $\bar{\pi}^l$ for the case of simple recourse. Go to step 2.

Some additional remarks about the algorithm are in order. The auxiliary minimization problem that is required in step 3 requires a considerable amount of computation.

An alternative convergence criterion is available at step 4. From results in [9] an upper bound on the difference between the minimum value of the objective function $\hat{z}(x^\circ)$ and the current value $\hat{z}(x^l)$ is given by

(12)
$$\hat{z}(x^l) - L_h,$$

where

$$L_h = \max_{h=1,\ldots,l} \hat{z}(x^h) + [c - \pi(x^h)T](\bar{x}^h - x^h).$$

Thus the problem may be terminated when (12) is less than some prespecified constant.

In addition, we note that a majority of the constraints in the second-stage problem of the monthly model, i.e., those of the form

$$\sum_{k \neq j} a_{ijk}x_{ijk} \leq a_{ij}x_{ij},$$

*The constraints may be reduced to this form without loss of generality by dividing each coefficient by the right-hand side and rescaling the variables.

and some of the first-stage constraints in the daily model are of a type known as generalized upper bounds [3]. These are constraints which appear schematically as

$$\begin{aligned} 1\ 1\ \ldots\ 1 \qquad\qquad\qquad\qquad &\leqq b \\ 1\ 1\ \ldots\ 1 \qquad\qquad\qquad &\leqq b_2 \\ \ddots \qquad\qquad\qquad &\ \vdots \\ 1\ 1\ \ldots\ 1 &\leqq b_2. \end{aligned}$$

That is, they are disjoint sums where the right-hand side is a nonnegative constant.* A special algorithm is available for solving problems of this form, which entails carrying a basis where size is reduced by L, the number of generalized upper-bound constraints.

Finally, as mentioned earlier, some rounding scheme is necessary to convert the numbers of assigned flights to integers. One scheme might be to first round up solutions whose fractional parts are 0.5 or greater and round down those whose fractional parts are less than 0.5. Then, if some flying hour constraint is exceeded, subtract one from a sufficient number of variables to satisfy that constraint. The variables to be reduced should be chosen from among those previously rounded up and may be those whose fractional part is closest to 0.5.

V. NUMERICAL EXAMPLE

We conclude with a small-scale numerical example of the solution of the monthly model. The problem consists of two aircraft types and two routes. The data were chosen for convenience. Solutions based on actual data will be reported in the future.

The data for the problem were as follows:

Aircraft type	Flying hours (round trip)		Carrying capacity (in tons)		Cost per flight (dollars)	
	Route 1	Route 2	Route 1	Route 2	Route 1	Route 2
1	24	14	50	75	7200	6000
2	49	29	20	20	7200	4000

Penalty cost ($ per ton)		
Route	Excess demand	Excess supply
1	500	0
2	250	0

In addition, the time to switch between either route was 5 hours for aircraft type 1 and 7 hours for aircraft type 2. The incremental costs for these moves were $1000 and $1500 for these aircraft types, respectively. It was assumed that 720 hours of flying time were available for each aircraft type. This can correspond to 3 planes of each type with an average utilization of 8 hours per day.

Finally, demand was assumed to be independently lognormally distributed on each route with the following parameters:*

Route	μ	σ
1	1000	50
2	1500	300

The following solution was found using an experimental program for the cutting-plane version of the algorithm.**

Aircraft type	Route	
	1	2
1	16.5	23.2
2	6.7	0.0

The total cost of the entire program converged after four iterations (based upon a termination when the relative decrease in costs fell below 1 percent). Each second-stage program was based upon drawing a sample of 25 observations so that, in effect, about 100 linear programming problems were solved during the course of the iterations. Execution time was 10 seconds.

For comparison purposes, a deteministic linear program was run with demand on routes assumed to be equal to the mean used in the lognormal case. This yielded the following results:

Aircraft type	Route	
	1	2
1	18.3	20.0
2	4.2	0.0

It should be noted that this solution differs from the case where the demands are stochastic. It might be supposed that the stochastic version would always assign more aircraft as a hedge against uncertainty; however, this is not the case even in this simple example—more flights of type 1 aircraft to route 1 are made in the deterministic version than in the stochastic.

---

*The current programs allow only for the lognormal distribution. It is a simple matter to add subroutines to generate random samples from arbitrary distributions whose cumulative distribution function can be tabulated. The lognormal distribution was chosen for experimental purposes because it produces nonnegative random variables which are skewed to the right—characteristics actual demand patterns are likely to exhibit.

**The program was written in FORTRAN and employs a small, all-in-core linear programming subroutine. Currently, the program is being modified to handle much larger problems and will be described in future work. A version of the algorithm for the monthly model using the gradient algorithm was also developed. Preliminary runs indicate that the gradient method takes considerably longer to run than the cutting-plane version because of the number of additional programs that must be solved during the auxiliary minimization problem described in Sec. IV.

ACKNOWLEDGMENTS

We wish to thank Dr. Roger Wets of the Boeing Scientific Research Laboratories for helpful discussions. We also wish to thank Lois Littleton of RAND, who wrote the program described in Sec. V and contributed a number of useful suggestions.

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* * *

A BRANCH-BOUND ALGORITHM FOR THE CAPACITATED FACILITIES LOCATION PROBLEM

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ABSTRACT

This paper discusses a mixed integer programming method for solving the Facilities Location Problem with capacities on the facilities. The algorithm uses a Decomposition technique to solve the dual of the associated continuous problem in each branch-bound iteration. The method was designed to produce the global optimum solution for problems with up to 100 facilities and 1,000 customers. Computational experience and a complete example are also presented in the appendix.

INTRODUCTION

The Facilities Location Problem involves determining from a predetermined finite number of potential sites the location and size of new facilities, i.e., warehouses, plants, and the like, so as to minimize total distribution costs. These costs consist of:

(1) the investment and operating costs of the facilities, and

(2) the inbound/outbound freight costs.

This problem cannot be solved by linear programming methods because of economies of scale associated with investment cost.

In the uncapacitated problem (no restriction on the size of facilities) while the freight costs are linear, the investment/operating (I/O) cost is usually considered continuous, piecewise linear, and concave, as in Fig. 1. Also, cost is zero if the facility is not built (no thruput).

It is shown in (3) that the uncapacitated problem is equivalent to the fixed cost problem, i.e.,

$$\text{Total I/O Cost}_i = \begin{cases} F_i > 0 & \text{Thruput}_i > 0 \\ 0 & \text{Otherwise} \end{cases}$$

and therefore the fixed cost problem is usually the only one considered.

For many problems, the uncapacitated formulation is adequate; however, for those cases where the facilities can only be built to a certain size or the cost curves "jump," i.e., where I/O cost curves have the form shown in Fig. 2, new methods are needed.

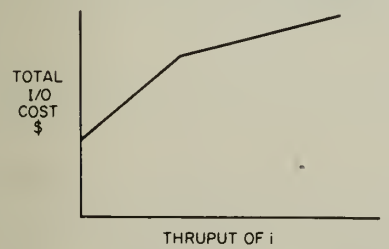


FIGURE 1. Potential Facility, i

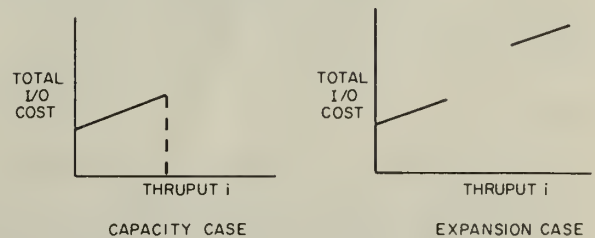


FIGURE 2. Potential Site, i

The uncapacitated problem has been studied extensively. "Heuristic" methods are explored in [5, 8, and 10]. Algorithmic methods are given in [4 and 7]. An excellent critique of approaches to this problem is given in [4] where the capacitated case is also studied. Also, applications of these techniques to planning problems are discussed in [1].

The Capacitated Problem

The Capacitated Facilities Location Problem with m plants (sources) and n customers (sinks) can be stated as

$$
\begin{aligned}
&\min Z = \sum_i \sum_j S_{ij}X_{ij} + \sum_i F_iY_i \\
&\sum_i X_{ij} = D_j && j = 1, \ldots, n \\
&\sum_j X_{ij} \leq A_i && i = 1, \ldots, m \qquad (1) \\
&x_{ij} - \min (D_j, A_i)Y_i \leq 0 && i = 1, \ldots, m; j = 1, \ldots, n \\
&Y_i = 0, 1 && i = 1, \ldots, m,
\end{aligned}
$$

where

S_{ij} = the unit transportation cost from plant i to customer j,

D_j = the demand at customer j,

$F_i \geq 0$ = the fixed cost associated with plant i,

A_i = the capacity at plant i,

X_{ij} = the quantity supplied to customer j from plant i,

$Y_i = 1 \Rightarrow$ the plant i is "open," and

$Y_i = 0 \Rightarrow$ the plant i is "closed".*

For many interesting practical problems, this formulation would give from ten to fifty-thousand rows and about the same number of columns. These problems are clearly too large for conventional mixed integer techniques unless the above linear problem can be solved at high speed. The algorithm given here employs the branch-bound technique developed by Land and Doig [9] and the Out-of-Kilter algorithm developed by Fulkerson [6] to solve the Capacitated Facilities Location Problem.

The Branch-Bound Algorithm

This paper discusses an efficient branch-bound † algorithm for solving the capacitated and, consequently, the jump problem. Computational experience for practical problems having about 50 potential sites and 300 demand points is also discussed.

The essence of the algorithm is to find the solution of a sequence of linear programs (1) where $Y_i (i = 1, \ldots, m)$ are individually treated either as continuous variables or are fixed to be 0 or 1. The branch-bound algorithm proceeds as follows:

Step I

The problem (1) is solved as a linear program ($Y_i (i = 1, \ldots, m)$ are treated as continuous vari-

* An extensive summary of Branch-Bound Methods is given in Ref. [2].

† It is easy to show that this formulation can be utilized when some of the facilities are intermediate points.

ables). If in the solution all Y_i are 0 or 1, then the mixed integer problem is solved.* If not, the value of $Z = Z_0$ in this solution is a lower bound on the solution to (1).

Termination with first linear program can come about because the "tight" formulation forces Y_i to be one if *any* customer's entire demand is supplied from plant i or if the entire capacity of plant i is shipped to a single customer.

Step II

If, in Step I, some Y_i is not integer we "branch" by fixing it first at 1 and then at 0 and re-solving (1) as a linear program. (In the algorithm the Y_i closest to 1/2 is selected to be fixed to 0 or 1). Clearly the lesser value of Z obtained is a new lower bound of the solution to (1). Also if all Y_i associated with this lower bound are integer then the problem is solved. Similarly, if there is no feasible solution when this Y_i is set to zero then plant i must be open in the final solution.

Step III

The algorithm continues "branching and bounding" until a solution to (1) is found that is the best lower bound developed.†

In utilizing Branch-Bound methods computational requirements dictate that a balance be drawn between cost (computer time) of calculating each bound and the number of branches that must be evaluated. In Ref. [4] the solution of the uncapacitated problem was characterized by a very rapid calculation of bounds and a rather large number of branches. For the algorithm discussed below, the "continuous" problem was "tightly" formulated and so relatively few branches had to be examined. In fact, for most practical problems analyzed the number of branches was extremely small.

Problem (1) can be restated as

$$
\begin{aligned}
&\min Z = \sum_{j=1}^{n} \sum_{i \epsilon N_j} S_{ij} X_{ij} + \sum^{m} F_i Y_i \\
&\sum_{i \epsilon N_j} X_{ij} \geq D_j && j = 1, \ldots, n \\
&-\sum_{j \epsilon P_i} X_{ij} \geq -A_i && i = 1, \ldots, m \\
&-X_{ij} + \min\,(D_j, A_i) Y_i \geq 0 && j = 1, \ldots, n;\ i \epsilon N_j \\
&Y_i = 0, 1 && i = 1, \ldots, m, \\
&X_{ij} \geq 0
\end{aligned}
\tag{2}
$$

where

N_j = list of plants which can supply customer j

P_i = list of customers which can be supplied by plant i.

The strategy for solving (2) is to solve (at each branch-bound iteration) the dual of the associated linear program by decomposition. The dual formulation of (2) is

---

*It is interesting to note that in almost all the practical problems solved using the algorithm, all Y_i were integer in the solution to the first linear program; however, it is easy to construct examples where this will not be the case. See Ref. [2].

†As will be shown, the dual of (1) is solved as linear program. Consequently, it is sometimes not necessary to carry each linear program to its solution to establish that an "integer" lower bound has been violated.

$$\max w = \sum_{j=1}^{n} D_j Q_j - \sum_{i=1}^{m} A_i C_i$$

$$\sum_{j \epsilon P_i} \min\,(D_j, A_i) U_{ij} \leqslant F_i \qquad i = 1, \ldots, m$$

$$Q_j - C_i - U_{ij} \leqslant S_{ij} \qquad j = 1, \ldots, n;\ i \epsilon N_j,$$

where Q_j, A_i, and U_{ij} are the dual variables.

By following the method given in Ref. [3], this problem "decomposes" naturally giving a "master problem"

(4) $$\sum_{j \epsilon P_i} \min\,(D_j, A_i) U_{ij} \leqslant F_i \qquad i = 1, \ldots, m$$

and a single "sub-problem" whose constraints are

(5) $$Q_j - C_i - U_{ij} \leqslant S_{ij} \qquad j = 1, \ldots, n;\ i \epsilon N_j;$$

and the objective function for the "sub-problem" at any decomp iteration is

(6) $$w_s = \sum_{j=1}^{n} D_j Q_j - \sum_{i=1}^{m} A_i C_i - \sum_{j=1}^{n} \sum_{i \epsilon N_j} \min\,(D_j, A_i) \pi_i U_{ij},$$

where the π_i are the dual variables of the extremal problem associated with (3). Furthermore, as is proved in the appendix, the optimality dual variables of the extremal problem are equal to the Y_i of (2).

The "sub-problem" can be quite large and cannot be solved easily in the form stated. However, the dual formulation of the "sub-problem" can be solved easily; and, the dual activities from its solution are the activities of the "sub-problem" at any decomposition iteration. The dual formulation of the "sub-problem" is

(7) $$\min \sum_{j=1}^{n} \sum_{i \epsilon N_j} S_{ij} X_{ij}$$

(7.1) $$\sum_{i \epsilon N_j} X_{ij} \geqslant D_j \qquad j = 1, \ldots, n$$

(7.2) $$-\sum_{j \epsilon P_i} X_{ij} \geqslant -A_i \qquad i = 1, \ldots, m$$

(7.3) $$-X_{ij} \geqslant -\min\,(D_j, A_i) \pi_i \qquad j = 1, \ldots, n;\ i \epsilon N_j$$

The dual activities of 7.1 are Q_j and the dual activities of 7.2 are C_i. Knowing the Q_j and C_i, the U_{ij} can be determined from

(8) $$U_{ij} = \max\,(0, Q_j - C_i - S_{ij}) \qquad j = 1, \ldots, n;\ i \epsilon N_j.$$

Relation (8) can be insured by the complementary slackness theorem [4]. If we rewrite (7) as

(9) $$\min \sum_{j=1}^{n} \sum_{i \epsilon N_j} S_{ij} X_{ij}$$

$$\sum_{i \epsilon N_j} X_{ij} = D_j \qquad j = 1, \ldots, n, \tag{9.1}$$

$$\sum_{j \epsilon P_i} X_{ij} \leq A_i \qquad i = 1, \ldots, m, \text{ and} \tag{9.2}$$

$$X_{ij} \leq \min (D_j, A_i)\pi_i \qquad j = 1, \ldots, n;\ i \epsilon N_j. \tag{9.3}$$

The dual activities of (9.1) are Q_j and the dual activities of (9.2) are $-C_i$. Problem (9) is a simple capacitated transportation problem. If we add a single dummy location ϕ to the physical problem described by (9), we can rewrite (9), as a network flow problem:

$$\sum_{i \epsilon N_j} X_{ij} - X_{j\phi} = 0 \qquad j = 1, \ldots, n, \tag{10.1}$$

$$X_{\phi i} - \sum_{j \epsilon P_i} X_{ij} = 0 \qquad i = 1, \ldots, n, \tag{10.2}$$

$$\sum_{j=1}^{n} X_{j\phi} - \sum_{i=1}^{m} X_{\phi i} = 0, \tag{10.3}$$

$$0 \leq X_{ij} \leq \min (D_j, A_i)\pi_i \qquad j = 1, \ldots, n;\ i \epsilon N_j, \tag{10.4}$$

$$0 \leq X_{\phi i} \leq A_i \qquad i = 1, \ldots, m, \tag{10.5}$$

$$D_j \leq X_{j\phi} \leq D_j \qquad j = 1, \ldots, n, \tag{10.6}$$

$$\min \sum_{j} \sum_{i \epsilon N_j} S_{ij} X_{ij}. \tag{10.7}$$

If the out-of-kilter algorithm is used to solve (10) the customer node prices, β_j, will be the dual activities of the Eqs. (10.1), the plant node prices, β_i, will be the dual activities of the Eqs. (10.2), and the node price at the dummy location, β_ϕ, will be the dual activity of (10.3).* The dual activities for (10.5) are $\beta_\phi - \beta_i$, and the dual activities for (10.6) are $\beta_j - \beta_\phi$. Hence,

$$\begin{aligned} Q_j &= \beta_j - \beta_\phi && j = 1, \ldots, n \\ C_i &= \beta_i - \beta_\phi && i = 1, \ldots, m \\ U_{ij} &= \max (0, \beta_j - \beta_i - S_{ij}) && j = 1, \ldots, n;\ i \epsilon N_j. \end{aligned} \tag{11}$$

Relation (11) can be used to generate extreme points for the extremal problem associated with (3) when (7) is feasible, i.e., when the "sub-problem" is bounded. When (7) is infeasible, however, the proper node prices in (10) may not be readily available when the out-of-kilter algorithm determines infeasibility. To overcome this difficulty, add an arc from the dummy location to each customer with a distribution cost of S^* where

$$S^* > 2 \sum_{i,j} (S_{ij}), \tag{12}$$

and with an upper bound equal to the demand at the customer. Thus, problem (10) becomes

$$\min \sum_{j} \sum_{i \epsilon N_j} S_{ij} X_{ij} + \sum_{j} S^* X_{\phi j}, \tag{13}$$

*Out-of-kilter is particularly effective here because it is simple to generate an almost optimal "start" for each decomposition iteration.

$$X_{\phi j}+\sum_{i\epsilon N_j} X_{ij}-X_{j\phi}=0 \qquad j=1, \ldots, n, \tag{13.1}$$

$$X_{\phi i}-\sum_{j\epsilon P_i} X_{ij}=0 \qquad i=1, \ldots, m, \tag{13.2}$$

$$\sum_{j=1}^{n} X_{j\phi}-\sum_{i=1}^{m} X_{\phi i}-\sum_{j=1}^{n} X_{\phi j}=0, \tag{13.3}$$

$$0 \leq X_{ij} \leq \min (D_j, A_i)\pi_i \qquad j=1, \ldots, n;\ i\epsilon N_j, \tag{13.4}$$

$$0 \leq X_{\phi i} \leq A_1 \qquad i=1, \ldots, m, \tag{13.5}$$

$$D_j \leq X_{j\phi} \leq D_j \qquad j=1, \ldots, n, \tag{13.6}$$

$$0 \leq X_{\phi j} \leq D_j \qquad j=1, \ldots, n. \tag{13.7}$$

The $X_{\phi j}$ arc will not have any flow in it unless customer j cannot be supplied otherwise. This fact permits us to generate the extreme ray as follows:

Step 1. If for any customer j, the arc $\beta_j-\beta_\phi > S^*/2$. Set Q_j to one, otherwise set Q_j to zero.

Step 2. If for any plant $i\beta_\phi-\beta_i > S^*/2$ set C_i to one, otherwise set C_i to zero.

Step 3. Compute the U_{ij} from

$$U_{ij}=\max (0, Q_j-C_i). \tag{14}$$

COMPUTATIONAL SUMMARY

1. Evaluate the unconstrained solution to (2). This is done by solving the extremal problem associated with the dual of (2), for its pi values by Decomposition. The sub-problem used to generate extreme points and extreme rays can be evaluated by solving (13) for its node prices by Out-of-Kilter.

2. In the unlikely event that the unconstrained solution found above is not all integer, choose a Y in (2) to force to 0 or 1 and solve (2) with $F_i=0$ if Y_i is set to 1, or $A_i=0$ if Y_i is set to 0.

3. If branch-bound optimality has not been reached, return to Step 2.

The algorithm has been implemented on a Burroughs B–5500 and has been tested on several medium-sized practical problems. These problems, formulated as in (1), possessed between 20 and 50 integer variables and between one and three-thousand constraints. In no problem was it necessary to make more than eight branches. The longest computer running time for any problem was 25 minutes; however, a more efficient computer program is under development that will cut running time by a factor of three.

Appendix I

THEOREM: Given the linear program

$$\begin{aligned} &\min Z=ax+by \\ &Ax+By \geq C, \end{aligned} \tag{1*}$$

where x is a vector of dimension m and y is a vector of dimension n, and the corresponding dual problem is represented as

$$(2^*) \qquad \begin{aligned} \max\ & w = UC \\ & A^T U \leq a \\ & B^T U \leq b \end{aligned}$$

If the dual problem is decomposed such that the master program is

$$(3^*) \qquad \begin{aligned} \max\ & w = UC \\ & A^T U \leq a, \end{aligned}$$

the sub-problem at any iteration is

$$(4^*) \qquad \begin{aligned} \max\ & w_s = (C - \pi A^T) U - \pi_0 \\ & B^T U \leq b \end{aligned}$$

where π is the vector of dual activities of the current extremal problem and π_0 is the dual activity associated with the convexity constraint in the extremal problem. Then, the optimal solution of (1*) requires that

$$(5^*) \qquad x = \pi^*,$$

where π^* is the vector of dual activities of the optimal extremal problem for Problem (2*).

PROOF: When the decomposition algorithm has converged to optimality, a vector U^* was obtained such that

$$(6^*) \qquad (C - \pi A^T) U^* \leq \pi_0.$$

The functional value of (2*) is then

$$(7^*) \qquad w^* = \pi^* a + \pi^* 0.$$

We can now show that $x = \pi^*$ permits Problem (1*) to have a feasible solution. To show this, assume that $x = \pi^*$ so that we have the problem

$$(8^*) \qquad \begin{aligned} \min\ & (a\pi^* + by) \\ & A\pi^* + By \geq C. \end{aligned}$$

We can show that there exists a vector y such that (8*) is feasible by examining the dual problem associated with it

$$(9^*) \qquad \begin{aligned} \max\ & (C - A\pi^*) U \\ & B^T U \leq b. \end{aligned}$$

Problem (9*) is bounded since relation (6*) is true, therefore (8*) has a feasible solution. Since the optimal solution to (1*) is always bounded below by w^*, Problem (8*) is also bounded.

We will now show that the solution to

(10*) $$\min \text{ by}$$
$$By \geq C - A\pi^*$$

is π_0^*. Let the vector y which gives the optimal solution to (10*) be represented by y^*, so that we need to show that $by^* = \pi_0^*$. $(C - \pi^* A^T)U^*$ must equal to π_0 since the left-hand side of (6*) is greater than or equal to $\pi^* + \pi_0^*$.

Therefore, the solution to (9*), which is the dual problem associated with (10*), is π_0^*. Since the primal and dual problems must have the same solution value, the solution to (10*) is also π_0^*. Hence, $\min by = \pi_0^*$ and the solution to (8*) is $\pi^*_a + \pi_0^*$ which is also the solution to (2*). It follows that the assumption that $x = \pi^*$ leads to the optimal solution of (1*) thereby proving the theorem.

Appendix II

EXAMPLE: This is a simple example involving two potential plants and two customers.

(I)

Plant (i)	Capacity (A_i)	Fixed Charge (F_i)
1	2	1
2	1	1

(II)

Customer (j)	Demand (D_j)
1	1
2	1

(III) Transportation Rates (S_{ij})

Plant (i) \ Customer (j)	1	2
1	3	1
2	0	1

(IV) The tableau for (1) is therefore

X_{11}	X_{12}	X_{21}	X_{22}	Y_1	Y_2		
3	1	0	1	1	1		
1		1				≥ 1	D_1
	1		1			≥ 1	D_2
1	1					≤ 2	A_1
		1	1			≤ 1	A_2
1				−1		≤ 0	
	1			−1		≤ 0	
		1			−1	≤ 0	
			1		−1	≤ 0	

(V) And the tableau for (3) is

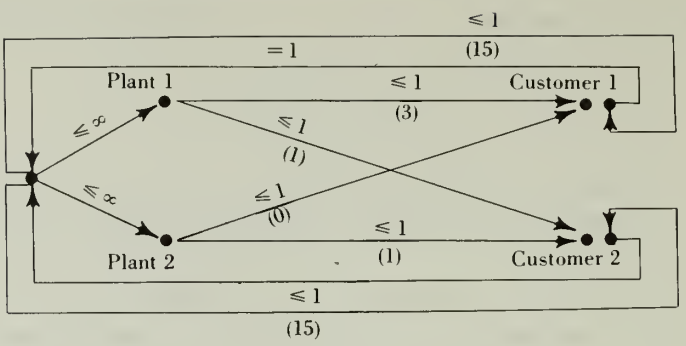
Q_1	Q_2	C_1	C_2	U_{11}	U_{12}	U_{21}	U_{22}		
1	1	−2	−1						
				1	1			≤ 1	F_1
						1	1	≤ 1	F_2
1		−1		−1				≤ 3	S_{11}
	1	−1			−1			≤ 1	S_{12}
1			−1			−1		≤ 0	S_{21}
	1		−1				−1	≤ 1	S_{22}.

(VI) The initial extremal problem is then

Slack 1	Slack 2	Artificial 1		
0	0	−10		
1			≤ 1	F_1
	1		≤ 1	F_2
		1	$=1$	Convexity Constraint.

Cost

(VII) We start the decomposition process by solving the uncapacitated allocation problem. This generates the following network problem



The solution to this problem (the uncapacitated transportation problem without regard to fixed costs) gives

$$X_{12}=1,\ X_{21}=1,\ \beta_{\text{Customer 1}}=0,\ \beta_{\text{Customer 2}}=1$$
$$\beta_b=\beta_{\text{Plant 1}}=\beta_{\text{Plant 2}}=0,$$

hence

$$\begin{aligned} Q_1&=0 & U_{11}&=0\\ Q_2&=1 & U_{12}&=0\\ C_1&=0 & U_{21}&=0\\ C_2&=0 & U_{22}&=0\\ \text{functional}&=1, \end{aligned}$$

so that the first extremal is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\0 \end{pmatrix}.$$

Since the network problem was feasible, the coefficient of the first extremal in the convexity constraint is 1 so that the new extremal problem becomes

Slack 1	Slack 2	Extremal 1		
0	0	1		Cost
1			≤ 1	F_1
	1		≤ 1	F_2
		1	$=1$	Convexity Constraint

which is merely the identity basis. As a result, the inverse is also the identity matrix so that the new pi values are

$$\pi_1 = 0$$
$$\pi_2 = 0$$
$$\pi_{\text{Convexity Constraint}} = 1.$$

It should be noted that if the first network problem is infeasible, then the current branch-bound iteration is also infeasible and we need not continue with the decomposition.

The solution to the network flow gives:

$$Q_1 = 1 \qquad U_{11} = 1$$
$$Q_2 = 1 \qquad U_{12} = 1$$
$$C_1 = 0 \qquad U_{21} = 1$$
$$C_2 = 0 \qquad U_{22} = 1$$
$$\text{functional} = 2,$$

so that the second candidate is

$$B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Since the network flow was infeasible, the coefficient of B in the convexity constraint is 0. Pivoting is in the second row so that the extremal problem becomes

Slack 1	Extremal 2	Extremal 1		
0	2	1		Cost
1	2		≤ 1	F_1
	2		≤ 1	F_2
		1	$=1$	Convexity Constraint.

(IX) The new inverse is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the new pi values are

$$\pi_1 = 0$$
$$\pi_2 = 1$$
$$\pi \text{ Convexity Constraint} = 1.$$

The solution to the network flow is therefore

$$Q_1 = 1 \qquad U_{11} = 1$$
$$Q_2 = 1 \qquad U_{12} = 1$$
$$C_1 = 0 \qquad U_{21} = 0$$
$$C_2 = 1 \qquad U_{22} = 0$$
$$\text{functional} = 1,$$

so that the third candidate is

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

This new extremal will pivot in the first row so that the new extremal problem is

Candidate 3	Candidate 2	Candidate 1		
1	2	1		Cost
2	2		≤ 1	F_1
0	2		≤ 1	F_2
		1	$=1$	Convexity Constraint.

(X) The new inverse is

$$\begin{pmatrix} 1/2 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the new pi values are

$$\pi_1 = 1/2$$
$$\pi_2 = 1/2$$
$$\pi \text{ Convexity Constraint} = 1.$$

The solution to the new network is

$$X_{11} = X_{12} = X_{21} = X_{22} = 1/2$$
$$\beta_{\text{Customer 1}} = 3 \qquad \beta_{\text{Customer 2}} = 1$$

or

$$Q_1 = 3 \qquad U_{11} = 0$$
$$Q_2 = 1 \qquad U_{12} = 0$$
$$C_1 = 0 \qquad U_{21} = 3$$
$$C_2 = 0 \qquad U_{22} = 0$$
$$\text{functional} = 2\ 1/2,$$

so that the fourth candidate is

$$\begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

The new candidate will pivot in row 2 so that the new extremal problem is

Candidate 3	Candidate 2	Candidate 1		
1	4	1		
2	0	0	≤ 1	Cost F_1
0	3	0	≤ 1	F_2
0	1	1	$=1$	Convexity Constraint.

(XI) The new inverse is

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix},$$

and the new pi values are

$$\pi_1 = 1/2$$
$$\pi_2 = 1$$
$$\pi \text{ Convexity Constraint} = 1,$$

and we are done.

The solution to this network is

$$X_{11} = X_{12} = X_{21} = X_{22} = 1/2$$
$$\beta_{\text{Customer 1}} = 3 \qquad \beta_{\text{Customer 2}} = 4 \qquad \beta_{\text{Plant 2}} = 3$$

or

$$Q_1 = 3 \qquad U_{11} = 0$$
$$Q_2 = 4 \qquad U_{12} = 3$$
$$C_1 = 0 \qquad U_{21} = 0$$
$$C_2 = 3 \qquad U_{22} = 0$$
$$\text{functional} = 2\ 1/2.$$

so the fifth candidate is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

The new candidate will pivot in the first row

Candidate 5	Candidate 2	Candidate 1		
4	4	1		
3	0	0	≤ 1	Cost F_1
0	3	0	≤ 1	F_2
1	1	1	$=1$	Convexity Constraint.

(XII) The new inverse is

$$\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ -1/3 & -1/3 & 1 \end{pmatrix},$$

and the new pi values are

$$\pi_1 = 1$$
$$\pi_2 = 1$$
$$\text{Convexity Constraint} = 1.$$

The solution to this is

$$X_{12} = X_{21} = 1$$
$$\beta_{\text{Customer 2}} = 1$$

or

$$Q_1 = 0 \qquad U_{11} = 0$$
$$Q_2 = 1 \qquad U_{12} = 0$$
$$C_1 = 0 \qquad U_{21} = 0$$
$$C_2 = 0 \qquad U_{22} = 0$$
$$\text{functional} = 1.$$

Since the functional equals the pi value of the convexity constraint and the new candidate is an extreme point and not an extreme ray the linear problem is optimized and the solution has $Y_1 = Y_2 = 1$ so both plants are opened. Since the linear solution is integer the mixed integer problem is solved.

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* * *

TIME-MINIMIZING TRANSPORTATION PROBLEMS

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ABSTRACT

A method is given for finding those solutions of a transportation problem which minimize the total time necessary for transporting goods from the suppliers to the consumers. Several extensions of the model are presented.

By a time-minimizing transportation problem we shall mean the problem of determining real non-negative numbers $x_{ij}(i=1, \ldots, m; j=1, \ldots, n)$ satisfying the constraints

$$\sum_{j=1}^{n} x_{ij}=a_i \qquad (i=1, \ldots, m) \tag{1}$$

and

$$\sum_{i=1}^{m} x_{ij}=b_j \qquad (j=1, \ldots, n), \tag{2}$$

where a_i and b_j are given non-negative numbers and

$$\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j,$$

and minimizing under these conditions the greatest of the given non-negative numbers t_{ij} for which $x_{ij}>0$.

From a practical point of view, the problem consists of supplying certain consumers with the necessary quantity of items (goods, military equipment, aircraft, and so on) in such a way that the total time from the starting of the operation until its completion, should be minimal. In this interpretation every supplier, i, has a given available amount, a_i, of items; every consumer, j, has a given necessity, b_j; and the time of transportation from i to j is t_{ij}.

Such a situation can arise for example, when military units are to be sent from their bases to certain fronts within the shortest possible time. Another obvious interpretation of this model concerns the transportation of perishable goods.

Analogously to the usual transportation problem, a set $X=(x_{ij})$ of non-negative numbers satisfying (1) and (2) will be called a feasible solution. A feasible solution for which

$$\max_{\{(i,j)\mid x_{ij}>0\}} t_{ij} \tag{3}$$

is minimal, will be called *almost-optimal*. The definition of basic (feasible) solutions is the same as in the case of common transportation problems.

Let $X^*=(x_{ij}^*)$ and $X^{**}=(x_{ij}^{**})$ be two feasible solutions of our problem; let

(4) $$M^*=\{(i,j)/x_{ij}^*>0\}$$

and

(5) $$M^{**}=\{(i,j)/x_{ij}^{**}>0\};$$

let

(6) $$t^*=\max_{(ij)\in M^*} t_{ij}$$

and

(7) $$t^{**}=\max_{(ij)\in M^{**}} t_{ij};$$

and let

(8) $$N^*=\{(ij)/t_{ij}=t^*\}$$

and

(9) $$N^{**}=\{(ij)/t_{ij}=t^{**}\}.$$

We shall say that the solution X^* is *better* than the solution X^{**}, either if $t^*<t^{**}$ or if $t^*=t^{**}$ and

$$\sum_{(ij)\in N^*} x_{ij}^* < \sum_{(ij)\in N^{**}} x_{ij}^{**}.$$

Two basic feasible solutions will be called *adjacent* if their bases differ in exactly one element.

A basic feasible solution will be called *locally optimal* if there is no better adjacent basic feasible solution.

A feasible solution will be called *optimal* if there is no better feasible solution. Obviously, any optimal solution is also an almost optimal solution.

In this paper, we shall be interested in determining only the optimal solutions of our problem.

It can be easily seen (as in the common transportation problem) that in seeking an optimal solution to our problem, we can restrict our attention to the basic feasible solutions. Moreover, we have the following:

THEOREM I: A feasible solution is optimal if and only if it is locally optimal.

PROOF: It is obvious that an optimal solution is also locally optimal.

Consider now a locally optimal solution $X^*=(x_{ij}^*)$ and suppose that a better solution, say $X^{**}=(x_{ij}^{**})$, exists.

Let M^*, M^{**}, t^*, t^{**}, N^*, N^{**} have the meanings given by (4), (5), (6), (7), (8), and (9), and let us put

(10) $$\sum_{(ij)\in N^*} x_{ij}^*=p^*$$

and

(11) $$\sum_{(ij)\in N^{**}} x_{ij}^{**}=p^{**}.$$

Let us define

(12) $$c_{ij}=\begin{cases} m+n & \text{if } t_{ij}>t^* \\ 1 & \text{if } t_{ij}=t^* \\ 0 & \text{if } t_{ij}<t^*, \end{cases}$$

and let us consider the cost-minimizing transportation problem defined by (1), (2), and by the cost matrix (c_{ij}). We see that

$$C^* = \sum\sum c_{ij}x_{ij}^* = t^*p^* > 0 \tag{13}$$

and

$$C^{**} = \sum\sum c_{ij}x_{ij}^{**} = t^{**}p^{**}. \tag{14}$$

If $t^{**} < t^*$, then $C^{**} = 0$, while if $t^* = t^{**}$ (and hence $p^{**} < p^*$) then $C^{**} = t^*p^{**}$. In both cases $C^{**} < C^*$, and hence X^* is not an optimal solution of the cost-minimizing transportation problem. It follows that an adjacent basic solution, say X^{***}, exists so that

$$C^{***} \leq C^*, \tag{15}$$

where

$$C^{***} = \sum\sum c_{ij}x_{ij}^{***}. \tag{16}$$

By defining M^{***}, N^{***}, t^{***}, and p^{***} as above, we can easily see that $t^{***} \leq t^*$, for otherwise $C^{***} > m+n$; while for the basic feasible solution X^*, obviously $C^* \leq m+n-1$, in contradiction with (15).

Let us suppose that $t^{***} = t^*$. Hence $C^{***} = p^{***}t^*$, while $C^* = p^*t^*$. From (15) it follows now that $p^{***} < p^*$.

Hence, either $t^{***} < t^*$, or $t^{***} = t^*$ and $p^{***} < p^*$. We conclude that the basic feasible solution X^{***}, adjacent to X^* is better than X^*, contradicting the fact that X^* is locally optimal.

From the fact that every time when step 2 is carried out, the value of p^*t^* is reduced at least by 1, it reached) by performing the following steps:

1. Determine a basic feasible solution;
2. Find an adjacent better basic feasible solution;
3. Perform step 2 until no adjacent basic feasible solution will be better than the considered one.

From the fact that every time step 2 is carried out, the value of p^*t^* is reduced at least by 1, it follows that the algorithm produces an optimal solution in a finite number of steps.

Step 1 can be carried out exactly as in the common transportation problem.

As to step 2, it consists in fact in the determination of an element (i, j) not in the basis, the introduction of which into the basis would reduce the value x_{hk} for at least one $(h, k) \in N$. As a matter of fact, we shall always determine that element (i, j) which introduced into the basis will reduce the value of the greatest x_{hk}, $\{(h, k) \in N\}$.

Hence, step 2 will consist of the following substeps:

2.1. Determine the greatest x_{hk}, with $(h, k) \in N$.

2.2. Determine the set $S_{h,k}$ of all elements (i, j) not in the basis, which if introduced into the basis would reduce the amount x_{hk}.

2.3. Choose among the elements of $S_{h,k}$ that one, say (h_0, k_0), for which $t_{h_0k_0}$ is the smallest.

2.4. Introduce $t_{h_0k_0}$ into the basis (as in the common transportation problem).

We see that the only substep which is difficult is 2–2. In order to solve this substep we need some definitions.

Let X be a basic feasible solution of our problem, and let M be its basis. To the set M of basic

elements (considered as vertices) we can associate a graph by considering any couple of vertices situated in the same row or the same column, as being linked by an arc.

Let us divide the set M into several levels, $L_0, L_1, L_2, \ldots$, in the following way: The level, L_0, will consist only of the vertex (h, k) determined in step 2–1. The level, L_1, will consist of all those elements of M, which are linked by arcs to $(h, k) \in L_0$. The level, L_2, will consist of all those elements of M which are linked by an arc to an element of L_1, etc. The decomposition into levels is unique, due to the lack of cycles in the graph of M.

Let us now associate to every row i an index $u(i)$ in the following way: If the elements of M belonging to row i are $(i, j_1) \in L_{q1}$, $(i, j_2) \in L_{q2}$, . . ., and if $q(i)$ is the smallest of the indices $q_1, q_2, \ldots$, then

$$u(i) = \begin{cases} 0 \text{ if } q(i) \text{ is even} \\ 1 \text{ if } q(i) \text{ is odd.} \end{cases} \tag{17}$$

Analogously, let us associate to every column j an index $v(j)$ in the following way: If the elements of M belonging to column j are $(j, i_1) \in L_{s_1}$, $(j, i_2) \in L_{s_2}$, . . ., and if $s(j)$ is the smallest of the indices $s_1, s_2, \ldots$, then

$$v(j) = \begin{cases} 0 \text{ if } s(j) \text{ is even} \\ 1 \text{ if } s(j) \text{ is odd.} \end{cases} \tag{18}$$

Further, let us put for every $(i, j) \in M$,

$$w(i, j) = u(i) + v(j). \tag{19}$$

We have proved in [1] (see also [2]) the following:

THEOREM II: $S_{h,k} = \{(i, j) \mid (i, j) \notin M,\ w(i, j) = 0\}$.

In order to illustrate our procedure, let us consider an example, with four producers supplying 37, 22, 32, and 14 units, respectively, with six consumers demanding 15, 20, 15, 25, 20, and 10 units, respectively, and with the following data concerning the transportation times t_{ij}:

TABLE 1

25	30	20	40	45	35
30	25	20	30	40	20
40	20	40	35	45	22
25	24	50	27	30	25

We can construct an initial feasible solution X^1 using for instance the North-West procedure:

15	20	2				0
		13	9			0
			16	16		0
				4	10	1
1	1	1	1	0	0	

Here†, $t^{*1}=45$, $N^{*1}=\{(3, 5)\}$, hence $(h, k)=(3, 5)$ and the levels are:

$$L_0^1=\{(3, 5)\}$$
$$L_1^1=\{(3, 4), (4, 5)\}$$
$$L_2^1=\{(2, 4), (4, 6)\}$$
$$L_3^1=\{(2, 3)\}$$
$$L_4^1=\{(1, 3)\}$$
$$L_5^1=\{(1, 1), (1, 2)\}.$$

Hence, $u^1(1)=0$, $u^1(2)=0$, $u^1(3)=0$, $u^1(4)=1$, and $v^1(1)=1$, $v^1(2)=1$, $v^1(3)=1$, $v^1(4)=1$, $v^1(5)=0$, $v^1(6)=0$ (these values were also written on the sides of X^1).

We deduce that:

$$S_{3,5}^1=\{(1, 5), (1, 6), (2, 5), (2, 6), (3, 6)\}.$$

The smallest t_{ij} with $(i, j)\in S_{3,5}^1$ being $t_{26}=20$, we introduce it into the basis, obtaining the following solution X^2:

15	20	2				1
		13			9	1
			25	7		0
				13	1	1
0	0	0	1	0	0	

† The superior index 1 refers to solution X^1.

Here, again $t^{*2}=45$, $N^{*2}=\{(3, 5)\}$, hence $(h, k)=(3, 5)$, and the levels are:

$$L_0^2=\{(3, 5)\}$$
$$L_1^2=\{(3, 4), (4, 5)\}$$
$$L_2^2=\{(4, 6)\}$$
$$L_3^2=\{(2, 6)\}$$
$$L_4^2=\{(2, 3)\}$$
$$L_5^2=\{(1, 3)\}$$
$$L_6^2=\{(1, 1), (1, 2)\}.$$

Hence, $u^2(1), \ldots, u^2(4), v^2(1), \ldots, v^2(6)$ have the values listed on the sides of X^2. We see that

$$S_{35}^2=\{(3, 1), (3, 2), (3, 3), (3, 6)\}.$$

The smallest t_{ij} with $(ij)\epsilon S_{35}^2$ being $t_{32}=20$, we introduce it into the basis, obtaining the following solution X^3:

15	19	3				0
		12			10	0
	1		25	6		0
				14		1
1	1	1	1	0	1	

Here again $T^{*3}=45$, $N^{*2}=(3, 5)$, hence $(h, k)=(3, 5)$, and the levels are

$$L_0^3=\{(3, 5)\}$$
$$L_1^3=\{(3, 2,) (3, 4), (4, 5)\}$$
$$L_2^3=\{(1, 2)\}$$
$$L_3^3=\{(1, 1), (1, 3)\}$$
$$L_4^3=\{(2, 3)\}$$
$$L_5^3=\{(2, 6)\}$$

Hence, $u^3(1), \ldots, v^3(6)$ have the values listed on the sides of X^3. We see that

$$S_{35}^3=\{(1, 5), (2, 5)\}.$$

The smallest t_{ij} with $(i, j)\epsilon S_{35}^3$ being $t_{25}=40$, we introduce it into the basis, obtaining the following solution X^4:

15	13	9				0
		6		6	10	0
	7		25			0
				14		1
1	1	1	1	0	1	

Here $t^{*4}=40$, $N^{*4}=\{(2, 5)\}$, hence $(h, k)=(2, 5)$, and the levels are

$$L_0^4=\{(2, 5)$$
$$L_1^4=\{(2, 3), (2, 6), (4, 5)\}$$
$$L_2^4=\{(1, 3)\}$$
$$L_3^4=\{(1, 1), (1, 2)\}$$
$$L_4^4=\{(3, 2)\}$$
$$L_5^4=\{(3, 4)\}.$$

Hence, $u^4(1), \ldots, v^4(6)$ have the values listed on the sides of X^4. We see that

$$S_{25}^4=\{(1, 5, (3, 5)\},$$

but as both t_{15} and t_{35} are greater than t_{25}, we deduce that X^4 is the optimal solution of our problem.

In case we wish to obtain *all* the optimal solutions of our problem, we proceed as in the above proposed algorithm, until we find an optimal solution X^*. Due to Theorem 1, all the other optimal solutions (if any) can be now obtained by successively introducing into the basis those elements $(i, j)\epsilon M^*$ for which $t_{ij}\leqslant t^*$.

Let us assume we have solved the original prblem, and have found an optimal solution X^* of it. Two refinements of the above discussed problem may occur frequently. Problem I, requires the determination of that

(i) optimal solution

(ii) almost optimal solution

of the original problem for which:

(20) $$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

is minimal; here c_{ij} are given nonnegative numbers (which may represent for example transportation costs).

(i) can be solved by minimizing the sum

(21) $$\sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}y_{ij},$$

under the constraints

$$\sum_{j=1}^{n} y_{ij} = a'_i \qquad (i = 1, \ldots, m) \tag{22}$$

$$\sum_{i=1}^{m} y_{ij} = b'_j \qquad (j = 1, \ldots, n) \tag{23}$$

$$y_{ij} \geq 0, \tag{24}$$

where

$$c'_{ij} = \begin{cases} \infty & \text{if } t_{ij} \geq t^* \\ c_{ij} & \text{if } t_{ij} < t^*, \end{cases} \tag{25}$$

and where

$$a'_i = a_i - \sum_{j/(i,j)\in N^*} x_{ij} \tag{26}$$

$$b'_j = b_j - \sum_{i/(i,j)\in N^*} x_{ij}. \tag{27}$$

If $Y^* = (Y^*_{ij})$ is an optimal solution of this last (common transportation) problem, and if the value of (21) in Y^* is finite, then it is easy to notice that

$$x^{**}_{ij} = \begin{cases} y^*_{ij} & \text{if } (i,j) \notin N^* \\ x^*_{ij} & \text{if } (i,j) \in N^*, \end{cases} \tag{28}$$

will be an optimal solution of Problem I in case (i).

If the value of (21) in Y^* is not finite, then it follows that the optimal solution of Problem I in case (i) is simply X^*.

For solving Problem I in case (ii), we simply solve the common transportation problem with constraints (1) and (2) and with the objective function

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c''_{ij} x_{ij}, \tag{29}$$

where

$$c''_{ij} = \begin{cases} c_{ij} & \text{if } t_{ij} \leq t^* \\ \infty & \text{if } t_{ij} > t^*. \end{cases} \tag{30}$$

For example, if in the above discussed numerical illustration we consider the following values of c_{ij}:

TABLE 2

120	200	150	250	240	210
150	160	125	180	230	130
230	130	260	190	250	100
140	150	300	310	170	140

then the optimal solution of the problem defined by (21), (22), (23), (24), (25), (26), and (27) will be:

Y^1

15	7	15			
	5		1		10
	8				
				14	

hence the optimal solution (in case (i)), given by formula (28) is:

X^5

15	7	15			
	5		1	6	10
	8				
				14	

For X^5, $t^*=40$ (as for X^4), $p^*=6$ (as for X^4), but the value of the "cost" function (20) is 12,530 (as compared to the value 17,970 of (20) for X^4).

Problem II consists of the following. Let us consider two basic feasible solutions X^* and X^{**} of the original problem.

Let us write all the values t_{ij} with $(i, j)\epsilon M^*$ in an ordered sequence:

(31) $$t_{i_1j_1} \geq t_{i_2j_2} \geq t_{i_3j_3} \geq \ . \ . \ . \ \geq t_{i_{m+n-1}j_{m+n-1}}.$$

Let us order analogously the values t_{hk} with $(hk) \in M^{**}$:

$$t_{h_1k_1} \geq t_{h_2k_2} \geq t_{h_3k_3} \geq \ldots \geq t_{h_{m+n-1}k_{m+n-1}}. \tag{32}$$

Let all the different values in (31) be

$$t_1^* > t_2^* > \ldots > t_f^* \tag{33}$$

and let all the different values in (32) be

$$t_1^{**} > t_2^{**} > \ldots > t_g^{**}. \tag{34}$$

Let

$$p_i^* = \sum_{(hk) \in N_i^*} x_{hk}^*, \qquad \text{where } N_i^* = \{(hk) \in M^* \mid t_{hk} = t_i^*\} \tag{35}$$

and

$$p_i^{**} = \sum_{(hk) \in N_i^{**}} x_{hk}^{**}, \qquad \text{where } N_i^{**} = \{(hk) \in M^{**} \mid t_{hk} = t_i^{**}\}. \tag{36}$$

Let $I_i = \{1, 2, \ldots, i\}$ and $I_0 = \phi$.

Let ℓ be the greatest $e \in \{0, 1, 2, \ldots,\}$ so that $t_e^* = t_e^{**}$ and $p_e^* = p_e^{**}$.

For $\ell + 1$, one of the following four alternatives holds:

(a) $t_{\ell+1}^* > t_{\ell+1}^{**}$

(b) $t_{\ell+1}^* = t_{\ell+1}^{**}$, $\quad p_{\ell+1}^* > p_{\ell+1}^{**}$

(c) $t_{\ell+1}^* = t_{\ell+1}^{**}$, $\quad p_{\ell+1}^* < p_{\ell+1}^{**}$

(d) $t_{\ell+1}^* < t_{\ell+1}^{**}$.

If (a) or (b) holds we shall say that X^* is lexicographically better than X^{**}; if (c) or (d) holds we shall that X^{**} is lexicographically better than X^*.

The purpose of this part of our paper is to extend the method described in the first part, for the determination of lexicographically optimal solution (i.e., solutions with no other one being lexicographically better).

For this sake, we first determine an optimal solution X^I to our original problem. We apply now the transformations (26) and (27) and put

$$t'_{ij} = \begin{cases} t_{ij} & \text{if } t_{ij} < t^* \\ \infty & \text{if } t_{ij} \geq t^*. \end{cases}$$

For the new problem we find an optimal solution X^{II}, and so on.

It is easy to see that if for every X^H we define $t^H = \max_{x_{ij} > 0} t_{ij}$, and $N^H = \{(ij), x_{ij} > 0, t_{ij} = t^H\}$, then a lexicographically optimal solution $\tilde{X} = (\tilde{x}_{ij})$ can be obtained by putting

$$\tilde{x}_{ij} = \begin{cases} x_{ij}^H & \text{if there is an } H \text{ so that } (ij) \in N^H \\ 0 & \text{in the opposite case} \end{cases}$$

(this formula makes sense because the N^H's are pairwise disjoint).

EXAMPLE: Let us reconsider our first example. X^4 is the first optimal solution. Here $N^I = \{(2, 5)\}$ and $x_{25} = 6$.

We now consider the problem with $a_1'=37$, $a_2'=16$, $a_3'=32$, $a_4'=14$; $b_1'=15$, $b_2'=20$, $b_3'=15$, $b_4'=25$, $b_5'=14$, $b_6'=10$, and the following t_{ij}''s:

25	30	20	∞	∞	35
30	25	20	30	∞	20
∞	20	∞	35	∞	22
25	24	∞	27	30	25

An optimal solution X^6 to this problem is:

15	7	15			
			16		
	13		9		10
				14	

Here $N''=\{(34)\}$, and $x_{34}=9$.

We consider now the problem with $a_1''=37$, $a_2''=16$, $a_3''=23$, $a_4''=14$; $b_1''=15$, $b_2''=20$, $b_3''=15$, $b_4''=16$, $b_5''=14$, $b_6''=10$, and the following t_{ij}'''s:

25	30	20	∞	∞	∞
30	25	20	30	∞	20
∞	20	∞	∞	∞	22
25	24	∞	27	30	25

An optimal solution X^7 to this problem is:

15	7	15			
			16		
	13				10
				14	

Here $N^{III}=\{(1, 2), (2, 4), (4, 5)\}$ and $x_{12}=7$, $x_{24}=16$, $x_{45}=14$.

We now consider the problem with $a_1''' = 30$, $a_2''' = 0$, $a_3''' = 23$, $a_4''' = 0$; $b_1''' = 15$, $b_2''' = 13$, $b_3''' = 15$, $b_4''' = 0$, $b_5''' = 0$, $b_6''' = 10$, and the following t_{ij}''''s:

25	∞	20	∞	∞	∞
∞	25	20	∞	∞	20
∞	20	∞	∞	∞	22
25	24	∞	27	∞	25

An optimal solution X^8 to it is:

15		15			
	13				10

Here, $N^{IV}=\{(1, 1)\}$ with $x_{11}=15$.

We now consider the problem with $a_1^{iv}=15$, $a_2^{iv}=0$, $a_3^{iv}=23$, $a_4^{iv}=0$; $b_1^{iv}=0$, $b_2^{iv}=13$, $b_3^{iv}=15$, $b_4^{iv}=0$, $b_5^{iv}=0$, $b_6^{iv}=10$, and the following t_{ij}^{iv}'s:

∞	∞	20	∞	∞	∞
∞	∞	20	∞	∞	20
∞	20	∞	∞	∞	22
∞	24	∞	∞	∞	∞

An optimal solution X^9 to it is:

		15			
	13				10

Here, $N^V = \{(3, 6)\}$ with $x_{36} = 10$.

In the next problem all the t_{ij}'s are equal to 20 or to ∞, hence no improvement can be expected. Obviously, $N^{VI} = \{(1, 3), (3, 2)\}$ with $x_{13} = 15$, $x_{32} = 13$.

Hence, we see that the lexicographically optimal solution of our problem, is:

15	7	15			
			16	6	
	13		9		10
				14	

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STOCK CONTROL IN A MANY DEPOT SYSTEM

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ABSTRACT

The chief problems considered are: (1) In a parallel set of warehouses, how should stocks be allocated? (2) In a system consisting of a central warehouse and several subsidiary warehouses, how much stock should be carried in each? The demands may have known, or unknown, distribution functions. For problem (1), the i-th stock n_i should usually be allocated in proportion to the i-th demand m_i; in special cases, a significant improvement is embodied in the formula (N = total allocable stock)

$$n_i = \max\ (0,\ m_i + \frac{\sqrt{m_i}}{\Sigma\sqrt{m_i}}\ \{N - \Sigma m_i\}).$$

For problem (2), the article gives specific bounds on intensity of demand, for deciding when an item in low demand should be stocked at a central rather than a branch warehouse.

1. INTRODUCTION

A considerable amount of literature exists on the inventory problem. The problem of redistribution (reallocation, transportation) has also been widely studied [1, 2, 3, 10]. The references [1, 5, 8, 9] are most closely related to the aspects of the problem treated in this article.

2. STOCKING OF DEPOTS IN A MULTI-DEPOT SYSTEM

In this section, it is assumed that demand and penalty functions are given at each of several parallel depots. The problem we consider may be stated as follows. Allocate N items to these depots so the expected stock-out cost (after a round of demands at the depots) is a minimum.

The same problem can be viewed as a redistribution problem among the depots, provided transportation cost is zero. This point of view was taken by S. G. Allen [1], where he anticipated some of the results obtained below. It may be interesting to note that the general conditions given for the validity of some of our results are necessary as well as sufficient; previously published work had given only sufficient conditions.

For given demand functions, the optimal allocation depends critically on the penalty functions; however, when the penalty function has the form

$$p(u) = \begin{cases} 0, & u \leqq 0 \\ a + bu, & u > 0 \end{cases} \qquad (a \geqq 0,\ b > 0), \tag{1}$$

an approximation to an optimal allocation is easily obtainable in many cases. The exceptional cases that require special treatment are chiefly those in which the average demand at one or more depots is less than about 4, and the demand at another depot exceeds about 15. Even so, the case is not ex-

ceptional unless the constants in the penalty function have values ($a \gg b$) that arise only seldom in practice.

3. GENERAL CONDITIONS

There are special cases in which it is optimal to allocate the entire supply of N units to one of several depots in the same echelon even when the depots are identical; several examples are given below.

One such special case is that in which r depots respond to identical (deterministic) demand functions $\phi_i(\xi)$, where

$$\phi_i(\xi) = \begin{cases} 0, \epsilon \neq N \\ 1, \xi = N (i = 1, 2, \ldots, r), \end{cases}$$

and after a single round of r demands, a penalty $p (p > 0)$ is paid at each depot which experiences a shortage, regardless of the amount of the shortage. Another such usual case is that in which the penalty function is the same as the above, and the demand functions satisfy $\phi_i(\xi) \equiv \phi(\xi)$, and $\phi(\xi)$ is such that

$$\phi(0) \leqslant \phi(1) \leqslant \phi(2) \leqslant \ldots \leqslant \phi(N). \tag{2}$$

(However, if all the inequalities are, in fact, equalities, any allocation is optimal.)

[This is easily established by an inductive argument; the allocation $(N, 0, \ldots, 0)$ is better than the allocation $(N-1, 1, 0, \ldots, 0)$, whereas the allocation $(n_1+1, n_2-1, n_3, \ldots, n_r)$ is no worse than the allocation $(n_1, n_2, n_3, \ldots, n_r)$ if $n_1 \geqslant n_2 > 0$.] These examples are pathological in the sense that an optimal policy is to stock the item in one depot only. A multi-depot system for which such a policy is always optimal is not worthy of the name. Discussion of specific unusual cases is concluded with the following easily proved lemma, which gives sufficient condition that a (fictitious) multi-depot system be pathological.

LEMMA 1: $N = n_1 + n_2 + \ldots + n_r$ units are allocated among r depots, n_i units being allocated to the i-th depot. After the allocation, each depot experienced a single demand, the demand at the i-th depot being ξ_i, with distribution function $\phi_i(\xi_i)$. If ξ_i exceeds $n_i (\xi_i > n_i)$, a penalty $p_i(\xi_i - n_i) \geqslant 0$ is incurred, so that the total penalty is $p_1(\xi_1 - n_1) + p_2(\xi_2 - n_2) + \ldots + p_r(\xi_r - n_r)$.

If each of the functions

$$T_i(n) = \sum_{\xi_i = n}^{\infty} p_i(\xi_i - n)\phi_i(\xi_i)$$

is a concave function, then there is at least one optimal allocation which allocates all units to some one depot.

The proof of this lemma is inductive, but is ommitted since a pathological case has little direct practical interest. Perhaps the conclusion to draw from the preceding cases is that all of them involve possible unrealistic assumptions on the penalty function, or on the demand function, or both. A model in which the penalty for a stock-out at depot X is a constant $\$p$ (which does not change with the number of units short) is probably unrealistic. (It may well be that a single long-distance telephone call represents the only apparent out-of-pocket cost of a shortage, however great; but in a multi-depot system,

there are likely to be other inconveniences. Not only do these depend on the amount of the shortage, but they probably outweigh the fixed amount $\$p$ in the long run.)

A final example will make this even clearer. Suppose five units are to be distributed to two depots, each of which experiences a single demand, and suppose a penalty $\$p$ is paid for each stock-out. (The penalty is thus either \$0, $\$p$, or $\$2p$.) If the demand functions are Poisson with means 2 and 20, respectively, all five units should be allocated to the first depot. If this solution seems peculiar and at variance with intuition, it is the penalty assumption that is at fault.

This concludes our discussion of problems that have untenable solutions. The discussion below concerns penalty and demand functions more likely to occur in practice. The basis of the discussion is Lemma 2.

4. SPECIAL PROBLEMS—TWO DEPOTS

LEMMA 2: Let $N=n_1+n_2$ units be allocated to two depots, n_1 units to the first, and $n_2(=N-n_1)$ units to the second. Subsequent to the allocation, each depot experiences a single demand, These demands $[\xi_1, \xi_2]$ are stochastic, with distribution functions $\phi_1(\xi_1)$, $\phi_2(\xi_2)$. If each of the functions

$$T_1(n_1)=\sum_{\xi_1=n_1}^{\infty} p_1(\xi_1-n_1)\phi_1(\xi_1)$$

and

$$T_2(n_2)=\sum_{\xi_2=n_2}^{\infty} p_2(\xi_2-n_2)\phi_2(\xi_2)$$

is a convex function of its indicated argument, the total expected penalty

$$p(n_1)=T_1(n_1)+T_2(N-n_1) \tag{3}$$

is a convex function of n_1. (See Figure 1.)

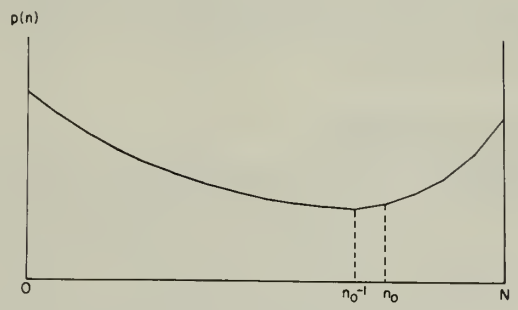


FIGURE 1. A typical function $p(n)$ of the type discussed in the test

PROOF: This lemma follows from the facts that (*i*) if $f(x)$ is convex on $[0, N]$, so if $f(N-x)$; and (ii) if $f(x)$ and $g(x)$ are convex on $[0, N]$, so is $f(x)+g(x)$.

COROLLARY 1: In the allocation problem described in Lemmas 1 and 2, if the penalty functions $p_1(x)$ and $p_2(x)$ have the forms $p_1(x)=\max(0, p_1x)$ and $p_2(x)=\max(0, p_2x)$, respectively, where p_1 and p_2 are positive constants, the total penalty $p(n_1)$ is necessarily a convex function of n.

This corollary is a consequence of the fact that $T_1(n_1)$ $[T_2(n_2)]$ is a convex function of the indicated argument.

This corollary, and some of the consequences of it given later, is due to S. G. Allen [1]. The case in which the penalty is a multiple of the number of units short is indeed one of the most important

cases. The problem is discussed in general context in [10], where, however, erroneous conclusions are reached. In fact, the optimal allocation need not occur of a value of n_1 interior to the interval $[0, N]$, but may occur at an end point of this interval. The type of correction which it is necessary to make in [10] is indicated by the hypotheses which are made in deriving results given later in this article. In particular, formula (5) of [10] should not be used when it brings out a negative value.

COROLLARY 2: If each function, $p_1(x)$ and $p_2(x)$, is convex on $[-\infty, \infty]$, the conclusion of Corollary 1 remains valid.

REMARK 2: If the penalty functions $p_1(x)$ and $p_2(x)$ have the more general form

$$p_i(x) = \begin{cases} p_{i0} + p_{i1}x, & x > 0 \\ 0, & x \leq 0, \end{cases} \qquad i = 1, 2, \tag{4}$$

with positive constants p_{i0}, p_{i1} $(i = 1, 2)$, direct evaluation of ΔT_1 shows that the functions $T_1(x)$ and $T_2(x)$ will be convex provided the hypotheses

$$\phi_i(0) \geq \phi_i(1) \geq \ldots \geq \phi_i(N) \tag{5}$$

are valid for $i = 1$ and 2. In fact $\Delta T_1 = -p_{10}\phi_1(n) - p_{11} \sum_{n+1}^{\infty} \phi_1(\xi)$; however if (5) does not hold, neither of the functions $T_1(x)$ nor $T_2(x)$, nor the total penalty $p(x)$ is necessarily a convex function.

If the demand function $\phi_i(\xi_i)$ has the Poisson form

$$\phi_i(\xi_i) = \frac{m^{\xi} e^{-m}}{\xi!}, \tag{6}$$

a brief exercise in algebra shows that if the mean demand m in (6) is less than or equal to $1 (m \leq 1)$, the total penalty is convex for all possible values of the positive constants p_{i0}, p_{i1}, since (5) is then valid. If $m > 1$, however, the function $T_i(n_i)$ is not a convex function unless the condition $p_{i0} \leq p_{i1}/(m_i - 1)$ holds, where m_i is the mean demand in (6). This assertion follows from the relation $\Delta^2 T_i = p_{i1}\phi(n+1) - p_{i0}[\phi(n+1) - \phi(n)]$.

5. ARBITRARY NUMBER OF DEPOTS

The problem above concerning two depots generalizes as follows. Suppose there are r depots, and suppose that $N = n_1 + n_2 + \ldots + n_r$ units are to be allocated among them. Immediately after the allocation, each depot experiences a stochastic demand, and pays a penalty if the demand exceeds the stock. The problem is to minimize the expected value of the total penalty.

If the demand at the i-th depot is ξ_i, its distribution function is $\phi_i(\xi_i)$ and the penalty function (of the shortage argument x) is $p_i(x)$, the function $p(n_1, n_2, \ldots, n_r)$ to be minimized is given by the formula

$$p(n_1, n_2, \ldots, n_r) = T_1(n_1) + T_2(n_2) + \ldots + T_r(n_r), \tag{7}$$

where $T_i(x)$ has the form

$$T_i(x) = \sum_{\xi_i = x}^{\infty} p_i(\xi_i - n_i)\phi_i(\xi_i).$$

The problem of minimizing (7) subject to the constraints

$$n_1 + n_2 + \ldots + n_r = N,$$

$$0 \leq n_i$$

is not the most elementary type, and cannot be solved by simply differentiating; this is the fallacy in the proof (and result) in [10]. Indeed the condition given in [10, p. 805] is never necessary, as claimed there; but if each $T_i(x)$ is a convex function, the condition is sufficient. The true situation is stated in Lemma 3.

LEMMA 3: Consider the allocation problem described in the first paragraph of this section, and let the functions $T_i(x)$ be convex on $[0, N]$. Suppose the allocation which distributes

$$n_i \text{ units to the } i\text{-th depot}, (i=1,2,\ldots,r)$$

has the property that no reallocation which alters the stocks in only two depots can decrease the expected total penalty. Then this allocation is optimal.

In other words, an allocation is optimal if it passes $r(r-1)/2$ tests of optimality.

By Lemma 2, each such test will take the simple form: Can an exchange of *one unit* between any pair of depots decrease the expected penalty? If not, the allocation is optimal.

PROOF OF LEMMA 3: The proof of Lemma 3 is by induction. As an induction hypothesis, it is assumed that the lemma is true for every reallocation which can be obtained from the given allocation (of n_i units to the i-th depot) by reallocating a total of not more than L units and prove the validity of the lemma for a reallocation of precisely $L+1$ units. Since the hypothesis assures the validity of the lemma under a reallocation of a single unit (only two depots can be involved in such a case), the truth of the lemma will follow.

To make the notation secure, suppose that the only depots involved in the reallocation are the first v depots, and that the allocation and reallocation are as shown:

	Depot Number						
	1	2	. . . .	u	$u+1$	. . . .	v
(A) allocation	n_1	n_2	. . . .	n_u	n_{u+1}	. . . .	n_v
(B) reallocation	n_1+k_1	n_2+k_2	. . . .	n_u+k_u	$n_{u+1}-l_{u+1}$	. . . .	n_v-l_v

where each of the numbers $k_1, \ldots, k_u, l_{u+1}, \ldots, l_v$ is positive. Let $EC(A)$ be the expected cost of allocation A. To show that the difference $EC(B)-EC(A)$ is nonnegative, it is sufficient to show that $EC(B)-EC(C)$ is nonnegative, where C is the reallocation:

(C) reallocation	n_1+k_1	n_2+k_2	n_u+k_u-1	$n_{u+1}-l_{u+1}$	n_v-l_v+1

for which it is known (induction hypothesis) that $EC(C)-EC(A)$. This is part of the induction hypothesis because $k_u-1 \geq 0$ and $k_1+k_2+\ldots+k_u-1=L$.

The value of $EC(B)-EC(C)$ is

$$EC(B)-EC(C)=T_u(n_u+k_u)-T_u(n_u+k_u-1)+T_v(n_v-l_v)-T_v(n_v-l_v+1). \tag{8}$$

Because of the assumed convexity of the functions T_u and T_v, the inequalities

$$\begin{aligned} T_u(n_u+k_u)-T_u(n_u+k_u-1) &\geq T_u(n_u+1)-T_u(n_u) \\ T_v(n_v)-T_v(n_v-1) &\geq T_v(n_v-l_v+1)-T_v(n_v-l_v) \end{aligned} \tag{9}$$

hold. From (8) and (9), the relation

$$EC(B)-EC(C) \geq T_u(n_u+1)-T_u(n_u)-[T_v(n_v)-T_v(n_v-1)] \tag{10}$$

follows. Since the right number of (10) represents the difference between the expected values of the penalties in (i) allocation (A) and (ii) that reallocation which differs from allocation A only in the reassignment of a single unit *from* depot u *to* depot v, and since this difference is nonnegative by hypothesis, the lemma is proved.

COROLLARY 1: Suppose $N=n_1+\ .\ .\ .\ +n_r$ units are to be allocated to r depots, n_i units to the i-th depot. Each depot experiences a demand, the demand ξ_i at the i-th depot being described by the distribution function $\phi_i(\xi_i)$. Each depot then pays a penalty $p_i(\xi_i-n_i)$ if the demand exceeds the stock, the penalty function being proportional to the units of shortage, i.e., $p_i(x)=\max(0, p_ix)$, where p_i are nonnegative constants. A sufficient condition for the total penalty

$$p(n_1, n_2, \ .\ .\ ., n_r)=p_1(\xi_1-n_1)+\ .\ .\ .+p_r(\xi_r-n_r)$$

to be a minimum is that the $r(r-1)/2$ equalities

$$p_i\int_{n_i}^{\infty}\phi(\xi_i)d\xi_i\approx p_j\int_{n_j}^{\infty}\phi(\xi_j)d\xi_j$$

hold.

[These are written as approximate equalities since the n_i are discrete variables. The meaning is that interchanges of a single unit would change the $\approx$ relation from $>$ to $<$.]

REMARK: These conditions were obtained in [10], but see [11].

PROOF OF COROLLARY: The corollary follows directly from the fact that for the penalty functions hypothesized, the functions $T_i(n_i)$ of Lemma 3 are convex. Thus for fixed n_i+n_j, the function

$$p(n_i)=T(n_i)+T(n_j),$$

is a convex function of n_i, and has a minimum value when its tangent line is horizontal (or changes slope from negative to positive).

A discussion of the conditions of Lemmas 1, 2, and 3 is in order. The main question is whether convexity is a natural condition in Lemmas 2 and 3 (i.e., one that often occurs in practical problems). In a certain sense, it is. To see this, suppose that $p_1(x)$ and $\phi_1(\xi_1)$ are penalty and demand functions so defined that the function

$$T_1(x)=\sum_{\xi_1=x}^{\infty}p_1(\xi_1-x)\phi_1(\xi_1)$$

fails to be convex for a single value x_0:

$$T_1(x_0+2)-T_1(x_0+1)<T_1(x_0+1)-T_1(x_0).$$

Then there is a penalty function $p_2(x)$ such that the total expected penalty

$$p(n)=T_1(n)+T_2(N-n)$$

in Lemma 2 fails to be even weakly convex for at least one value of n (namely $n=x_0$), no matter what the demand function $\phi_2(\xi_2)$ may be. This assertion is verified by setting $p_2(x)\equiv 0$; for then, $T_2\equiv 0$.

Thus the condition that $T_i(x)$ be convex is almost necessary: it is necessary in the sense that no

less restrictive condition on the situation in the i-th warehouse will be sufficient for the convexity of $p(n)$ *for every possible situation at the other warehouses.* In this sense, the condition is both necessary and sufficient

6. CONGRUENT WAREHOUSES

We call two warehouses congruent if their demand functions are equal.

COROLLARY 2: If all demand functions on Corollary 1 are equal

$$\phi_1(\xi) = \phi_2(\xi) = \ldots = \phi_r(\xi);$$

an optimal allocation is one in which no two stocks differ by more than one unit.

This corollary is a direct consequence of Lemma 3.

It does not follow from this statement that congruent depots should always be treated equally; the pathological cases described on pp. 3–5 furnish counterexamples to this assertion (and incidentally to the assertion on lines 20–23 on p. 800 of [10]).

7. UNKNOWN DEMAND FUNCTIONS

Lemma 3 can be generalized to the case of unknown distribution of demand (so called in [6]). In this case, the demand functions are not known explicitly, but each one is known to belong to a certain family of demand functions, the i-th one to the family $\mathcal{F}_i$. In this more general problem, an optimal allocation is defined as an allocation such that the supremum of the expected cost over the set of demand functions is as small as possible. (This type of allocation is sometimes called a minimax allocation.)

In the two-depot case, the situation is as follows. There is given a family $\mathcal{F}_1 = \{\phi_1^\alpha(\xi)\}$ of demand functions $\phi_1^\alpha(\xi)$, and the actual demand function at the first depot, though not known explicitly, is known to be a member of this family. The superscript α is an index that counts the members of the family. A parallel situation exists at the second depot. Since either or both of the families $\mathcal{F}_1, \mathcal{F}_2$ may contain just a single demand function, the situation about to be treated does indeed include, and is a generalization of, the situation already considered.

The penalty function which generalizes formula (3) in this case is

$$p^*(n) = T_1^*(n) + T_2^*(N-n), \tag{11}$$

where

$$T_i^*(n) = \sup_\alpha T_i^\alpha(n),$$

$$T_i^\alpha(n) = \sum_{\xi=n}^{\infty} p_i(\xi - n)\phi_i^\alpha(\xi), \qquad i = 1, 2,$$

the sup being taken over all demand functions in the family 1 or 2, as the case may be.

The following lemma gives a somewhat weaker condition that $r(r-1)/2$ tests of optimality are sufficient to insure that an allocation of N objects among r depots is optimal. The proof of this lemma does not differ substantially from the proof of Lemma 3.

LEMMA 3a: Let N objects be allocated among r depots, n_i units to the i-th depot. Suppose the terms $T_i(n_i)$ in the expected penalty function $\Sigma_i T_i(n_i)$ have the property that each of the $r(r-1)/2$ sums

$$T_i(n_i) + T_j(n_j) \qquad (i \neq j)$$

of pairs of these terms is a convex function of n_i (for fixed n_i+n_j with $0 \leq n_i+n_j \leq N$). Then if no exchange between two depots can reduce the expected penalty, the allocation $\{n_1, n_2, \ldots, n_r\}$ is optimal.

This lemma states that if an allocation passes $r(r-1)/2$ tests of optimality, and if the functions $T_i(n_i)$ have weak convexity properties, the allocation is optimal. Even with the stronger hypothesis of Lemma 3 (that each $T_i(n_i)$ is convex), the number $r(r-1)/2$ cannot be reduced.

It is clear that $T_1(n_1)+T_2(\bar{N}-n_1)$ can be a convex function of n_1 on $[0, \bar{N}]$ for all $\bar{N}$, $0 \leq \bar{N} \leq N$ even though $T_1(n_1)$ is not itself convex on $[0, N]$.

LEMMA 4: Let $\{\nu\}$ be an index set, and let $\{T^\nu(x)\}$ be any family of convex functions defined for $0 \leq x \leq N$. Then if the function $T^*(x) = \sup_\nu T^\nu(x)$ exists, this function is also convex.

A formal proof of this lemma is easily constructed from the definition of "convex function."

LEMMA 5: Let families $\mathcal{F}_1$, $\mathcal{F}_2$ of demand functions be given at each of two depots; suppose that the penalty function $p_1(x)$ has the property that for every $\phi_1^\alpha(\xi) \in \mathcal{F}_1$,

$$T_1^\alpha(n) = \sum_{\xi=n}^{\infty} p_1(\xi-n)\phi_1^\alpha(\xi)$$

is a convex function of n, and that the penalty function $p_2(x)$ has the corresponding property with $T_2^\beta(n)$, $\phi_2^\beta(n)$ in place of $T_1^\alpha(n)$, $p_1^\alpha(n)$. Then the function $p^*(n)$ defined by (11) is a convex function of n.

This lemma follows immediately from Lemmas 3, 4.

Now let there be $r(\geq 2)$ depots, and a family of demand functions $\mathcal{F}_i = \{\phi_i^\alpha(\xi)\}$ defined at the i-th depot ($i=1, 2, \ldots, r$). If $p_i(x)$ is the penalty function (cost of a stock-out of x units) at the i-th depot, the total expected stock-out cost $p(n_1, n_2, \ldots, n_r; \alpha, \beta, \ldots, \zeta)$ for fixed demand functions $\phi_1^\alpha(\xi)$, $\phi_2^\beta(\xi), \ldots, \phi_r^\zeta(\xi)$ and for the allocation which allots n_i units to the i-th depot is defined by the relation

$$p(n_1, n_2, \ldots, n_r; \alpha, \beta, \ldots, \zeta) = T_1^\alpha(n_1) + T_2^\beta(n_2) + \ldots + T_r^\zeta(n_r),$$

where

$$T_i^\nu(n_i) = \sum_{\xi=n_i}^{\infty} p_i(\xi-n_i)\phi_i^\nu(\xi), \qquad i=1, 2, \ldots, r.$$

The function $p^*(n_1, n_2, \ldots, n_r)$ to be minimized is defined by the relation

$$p^*(n_1, n_2, \ldots, n_r) = \sup_{\alpha\beta\ldots\zeta} p(n_1, n_2, \ldots, n_r; \alpha, \beta, \ldots, \xi).$$

But this is clearly equivalent to the formulation

$$p^*(n_1, n_2, \ldots, n_r) = T_1^*(n_1) + T_2^*(n_2) + \ldots + T_r^*(n_r),$$

with (see note at end of this section)

$$T_i^*(n_i) = \sup_\nu T_i^\nu(n_i), \qquad i=1, 2, \ldots, r.$$

Thus each of the functions $T_i^*(n_i)$ is convex, and the situation already looks very much like that in Lemma 3. In fact, the proof of Lemma 3 used nothing but the convexity of the individual summands $T_i(n_i)$, so the following lemma holds.

LEMMA 6: (Generalization of Lemma 3.) Let a family $\mathcal{F}_i = \{\phi_i^\alpha(\xi)\}$ $(i = 1, 2, \ldots, r)$ of demand functions be given at each of r depots, and let a penalty function $p_i(x)$ be defined at the i-th depot $(i = 1, 2, \ldots, r)$. Suppose further that each of the functions

$$T_i(x) = \sum_{\xi=x}^{\infty} p_i(\xi - x)\phi_i^\alpha(\xi)$$

is convex (for every member ϕ_i^α of $\mathcal{F}_i$). [This condition holds automatically if $p_i(x) = \max(0, p_i \cdot x)$.] When $N = n_1 + \ldots + n_r$ units are allocated among the r depots, n_i to the i-th depot, the expected stock-out cost is the quantity

$$p(n_1, \ldots, n_r; \alpha, \beta, \ldots, \zeta) = T_1^\alpha(n_1) + T_2(n_2) + \ldots + T_r^\zeta(n_r), \tag{12}$$

if the demand functions are known to be $\phi_1^\alpha(\xi)$, $\phi_2^\beta(\xi)$, $\ldots$, $\phi_r^\zeta(\xi)$ at the respective depots.

If the quantity

$$p(n_1, \ldots, n_r) = \sup_{\alpha, \beta, \ldots, \zeta} p(n_1, \ldots, n_r; \alpha, \beta, \ldots, \zeta)$$

cannot be decreased by exchanges of a single item between any two depots, the allocation $[n_1, \ldots, n_r]$ is optimal.

EXPLANATION OF TAKING SUPREMUM: The strategy that minimizes $T_i^*(n_i)$ is familiar in game theory; it is a strategy to minimize losses against the most damaging play the opponent can make.

8. ALLOCATION AMONG SEVERAL DEPOTS WHEN THE STOCK-OUT PENALTY HAS THE FORM $\alpha + \beta s$

Suppose $N = n_1 + n_2 + \ldots + n_r$ units are to be allocated among r depots subject to demands from independent consumers; the problem is to minimize the expected cost of stock-outs. If the cost of a stock-out at the i-th depot is $\beta_i s$, i.e., is proportional to the number of units backordered, the cost functions are convex and the problem is tractable. If further the β_i are all equal:

$$\beta_1 = \beta_2 = \ldots = \beta_r,$$

a reasonably satisfactory analytic solution is to take n_i to be

$$n_i = \max\left[0, m_i + \frac{\sigma_i}{\Sigma\sigma_i}(N - \Sigma m_i)\right], \tag{13}$$

where m_i is the mean and σ_i is the standard deviation of the demand at the i-th depot. The formula (13) is only approximate, not exact as claimed in [10]; its validity is discussed below.

If the stock-out cost is not of the form $\beta_i s$ at each depot, however, the cost functions need not be convex, and the only rigorous method of finding an optimal allocation is to compare the expected costs of all r^N possible allocations. Even when N is 2 or 3, this is not a reasonable procedure in the operation of a supply system that handles thousands of items. Fortunately an alternative method can be given which yields results that are nearly optimal in many cases; in fact the exceptional cases may rarely arise in a practical situation.

The observation which lies at the basis of the method is the assertion that for small N, many of the multifarious possible allocations are equally potent in minimizing the expected number of depots

which run out of stock. This statement can be verified by direct calculation when the demand function is Poisson. Table 1 below shows the expected number of stock-outs when N units are allocated to two depots with Poisson demands having equal means ($m_1 = m_2$). The allocation which splits the stock evenly is close to optimal.

TABLE 1

$(m_1 = m_2)$	N	Allocation		Expected number of depot stock-outs	
		n_1	n_2	maximum	minimum
1	2	0	2	0.712	
		1	1		0.528
1	3	0	3	0.651	
		1	2		0.344
2	2	0	2	1.187	
		1	1		1.187
	3	0	3	1.007	
		1	2		0.917
	4	0	4	0.917	
		2	2		0.646
	6	0	6	0.869	
		3	3		.285
3	3	0	3		1.303
		1	2	1.377	
	6	0	6	0.984	
		3	3		0.705
5	5	0	5		1.377
		2	3	1.610	
	10	0	10	1.006	
		5	5		0.768
10	10	0	10		1.416
		5	5	1.865	
	20	0	20	1.001	
		10	10		0.833
20	5	0	5	2.	
		2	3		2.
	20	0	20		1.440
		10	10	1.978	
	40	0	40	1.	
		20	20		0.881

If the mean demands at the two depots are far from equal, the above statement is not valid for very small values of N, the total stock. This is an unpleasant property of the penalty function α being considered (number of depots out of stock, independent of number of units backordered).

In real life, the penalty function is likely to have the form $\alpha + \beta s$, that is, to be the sum of two penalties: $\$\alpha$ for each depot unable to meet its demands, and $\$\beta$ for each unit which must be backordered. $\$\alpha$ may be a set-up cost for ordering, and $\$\beta$ may represent the actual cost of each unit of shortage.

In view of the above facts, a good approximation to the optimal allocation in many practical cases can be obtained by using the optimal allocation for the case in which the penalty has the form βs. This case is the subject of the next section. It would be risky, however, to ignore set-up costs without further study in a particular situation.

9. GENERAL STOCKING POLICY

There are circumstances in which it is advantageous to stock several depots with various quantities of the same item. In this section we assume the demand function is known at each depot, and ask how to allocate a fixed number of units among the depots. In this simplified model, it is assumed not to be advantageous to stock a central warehouse serving the depots—perhaps because no central warehouse exists. Transshipment between depots is not permitted; this is equivalent to the assumption that transshipment costs are at least as great as the cost of ordering items directly from the supplier.

Suppose first that there are two depots subject to stochastic demands m_1, $m_2 (m_1 \leqslant m_2)$. [Generalization to the case of r depots is given later.]

The penalty function is assumed to be a fixed charge per unit of shortage. Thus if n_1 units are allocated to depot 1 and n_2 units are allocated to depot 2, the penalty $p(\xi_1, \xi_2)$ to be paid is defined by

$$p(\xi_1, \xi_2) = p_1(\xi_1) + p_2(\xi_2),$$

where

$$p_1(\xi_1) = \begin{cases} 0, & \xi_1 \leqslant n_1 \\ b_1(\xi_1 - n_1), & \xi_1 > n_1, \end{cases}$$

and similarly for $p_2(\xi_2)$. Here ξ_1 and ξ_2 are the respective demands.

If the total number $n_1 + n_2$ of units to be stocked is fixed at N, it might be conjectured that the items should be allocated in the ratio $m_1 : m_2$, but this is not the case, even for Poisson demands. In fact, the allocation ratio varies with N, except in the case $m_1 = m_2$, i.e., the case in which the depots are identical.

Let the demand functions be $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$. The problem is to choose n_1 and n_2 so the expected penalty $\Sigma\Sigma p(\xi_1, \xi_2)\phi_1(\xi_i)\phi_2(\xi_2)$ is a minimum.

Denoting this expected penalty by $P(n_1, n_2)$ we find

$$P(n_1, n_2) = b_1 \sum_{\xi_1 = n+1}^{\infty} (\xi_1 - n_1)\phi_1(\xi_1) + b_2 \sum_{n_2+1}^{\infty} (\xi_2 - n_2)\phi_2(\xi_2),$$

so that the difference $P(n_1+1, n_2-1) - P(n_1, n_2)$ has the value

$$P(n_1+1, n_2-1) - P(n_1, n_2) = b_2 \sum_{n_2}^{\infty} \phi_2(\xi_2) - b_1 \sum_{n_1+1}^{\infty} \phi_1(\xi_1). \tag{14}$$

This difference is a monotonic function of n_1, i.e., the difference (14) is increased if n_1 is increased by one unit and n_2 is simultaneously decreased by one unit. If $P(n_1, n_2)$ is to be a minimum, the relations

$$\frac{\sum_{\xi_2 = n_2+1}^{\infty} \phi_2(\xi_2)}{\sum_{\xi_1 = n_1}^{\infty} \phi_1(\xi_1)} \leqslant \frac{b_1}{b_2} \leqslant \frac{\sum_{\xi_2 = n_2}^{\infty} \phi_2(\xi_2)}{\sum_{\xi_1 = n_1+1}^{\infty} \phi_1(\xi_1)} \tag{15}$$

must hold. We write $b_1/b_2=t$, and rewrite (15) as the two simultaneous inequalities

$$\sum_0^{n_2}\phi_2(\xi_2)\geq(1-t)+t\sum_0^{n_1-1}\phi_1(\xi_1), \tag{16}$$

$$\sum_0^{n_2-1}\phi_2(\xi_2)\leq(1-t)+t\sum_0^{n_1}\phi_1(\xi_1). \tag{17}$$

While (16) and (17) do determine n_1 and n_2, they are awkward to use. In case $b_1=b_2$ (i.e., in case $t=1$), a case which is of the greatest practical importance, a very simple approximate solution can be given if the demand functions are of Poisson type. This can be done by making use of the formula [7, p. 717]

$$\sum_0^x e^{-m}\frac{m^\xi}{\xi!}\approx\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-u^2}du,\ y=\frac{x+\frac{1}{2}-m}{\sqrt{m}}. \tag{18}$$

Thus if $t=1$, and the demands are of Poisson type with means m_1 and m_2, (16) and (17) become, respectively,

$$\frac{n_2+\frac{1}{2}-m_2}{\sqrt{m_2}}\geq\frac{n_1-\frac{1}{2}-m_1}{\sqrt{m_1}} \tag{19}$$

and

$$\frac{n_2-\frac{1}{2}-m_2}{\sqrt{m_2}}\leq\frac{n_1+\frac{1}{2}-m_1}{\sqrt{m_1}}. \tag{20}$$

Since n_1+n_2 is fixed, these inequalities, which are approximate circumlocutions for (16) and (17), require that n_1 and n_2 satisfy

$$\frac{n_2-m_2}{\sqrt{m_2}}=\frac{n_1-m_1}{\sqrt{m_1}} \tag{21}$$

as closely as that equality can be satisfied when n_1 and n_2 are integers; i.e., n_1 is the integer closest to the exact solution of

$$\frac{N-x-m_2}{\sqrt{m_2}}=\frac{x-m_1}{\sqrt{m_1}}.$$

Thus n_1 and n_2 are given approximately by the formulas

$$n_1=m_1+\frac{\sqrt{m_1}}{\sqrt{m_1}+\sqrt{m_2}}\ [N-m_1-m_2] \tag{22}$$

and

$$n_2 = m_2 + \frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}} \quad [N - m_1 - m_2]. \tag{23}$$

Limitations on the validity of these formulas are outlined later in this section.

The ratio n_1/n_2 has the approximate value

$$\frac{n_1}{n_2} \approx \frac{[N-m_2]\sqrt{m_1}+m_1\sqrt{m_2}}{[N-m_1]\sqrt{m_2}+m_2\sqrt{m_1}}. \tag{24}$$

When $m_1 = m_2$, (22) and (23) have the value $N/2$, and (24) has the value 1, all of which are exactly correct if N is an even integer. When m_1 and m_2 are unequal, the right member of (24) varies with N; however, relation (24) is only approximate, and it is *a priori* conceivable that the right member of (24) might vary strongly with N, while the left member which it supposedly approximates vary very little. Direct calculations do show, however, that the ratio n_1/n_2 of optimal allocations n_1 and n_2 at the two depots depends on N in a substantial way if the means m_1 and m_2 of the demands at these depots are widely different.

10. VALIDITY OF THE APPROXIMATIONS (22), (23)

A theoretical justification of the error of the approximation (18) [which led to relations (22) and (23)] does not appear in [7]. Numerical verification has established that, in some cases at least (22) and (23) are quite accurate up to $N = 3(m_1 + m_2)$. On the other hand, (22) can be negative for small values of N; thus (22) and (23) should not be used unless N satisfies the inequality.

$$N \geq m_2 - \sqrt{m_1 m_2}, \tag{25}$$

i.e., unless (22) brings out a positive value n_1. Moreover, (22) and (23) may not be wholly reliable even when N barely satisfies (25). At least, (22) and (23) may be useful first approximations to the quantities n_1 and n_2 which they purport to evaluate.

11. THE MANY-DEPOT CASE

It is possible to analyze a stocking problem for an arbitrary number of depots on the basis of the above results.

The problem is as follows. There are r depots, each of which is subject to a stochastic demand, the demand function $\phi_i(\xi)$ at the i-th depot being given by the Poisson formula

$$\phi_i(\xi_i) = \frac{e^{-m_i} m_i^{\xi_i}}{\xi_i!}, \quad (i = 1, 2, \ldots, r),$$

in which the mean value of the demand is m_i.

A fixed number $N = n_1 + \ldots + n_r$ of units is to be allocated, n_i to the i-th depot ($i = 1, 2, \ldots, r$), so that the expected shortage penalty P is a minimum. This expected penalty is computed from the formula

$$P(n_1, n_2, \ldots, n_r) = P_1(n_1) + \ldots + P_r(n_r),$$

where the expected penalty $P_i(n_i)$ at the i-th depot is the weighted sum

$$P_i(n_i) = \sum_{\xi_i = n_i + 1}^{\infty} (\xi_i - n_i)\phi_i(\xi_i).$$

Thus the actual shortage penalty at the i-th depot is simply k dollars if the demand at the depot exceeds the stock by k units.

As will be shown in the discussion immediately following, an approximate solution to this problem (a close-to-optimal allocation) is to determine n_i according to the formula

$$n_i = m_i + \frac{\sqrt{m_i}}{\Sigma\sqrt{m_\nu}}\{N - \Sigma m_\nu\}. \tag{26}$$

[This formula should not be used, however, when it imputes a negative value to any of the n_i.]

The conclusion immediately above is arrived at by making use of Lemma 3. This lemma asserts that in the case at hand, if an allocation is optimal with respect to exchanges between pairs of depots (i.e., if the allocation passes $r(r-1)/2$ tests of optimality) then the allocation is (totally) optimal. Thus (as an illustration) if there are three depots, the quantities n_1, n_2, and n_3 must satisfy the six relations

$$\frac{n_2 + \frac{1}{2} - m_2}{\sqrt{m_2}} \geqslant \frac{n_1 - \frac{1}{2} - m_1}{\sqrt{m_1}},$$

$$\frac{n_2 - \frac{1}{2} - m_2}{\sqrt{m_2}} \leqslant \frac{n_1 + \frac{1}{2} - m_1}{\sqrt{m_1}},$$

$$\frac{n_3 + \frac{1}{2} - m_3}{\sqrt{m_3}} \geqslant \frac{n_2 - \frac{1}{2} - m_2}{\sqrt{m_2}},$$

$$\frac{n_3 - \frac{1}{2} - m_3}{\sqrt{m_3}} \leqslant \frac{n_2 - \frac{1}{2} - m_2}{\sqrt{m_2}},$$

$$\frac{n_1 + \frac{1}{2} - m_1}{\sqrt{m_1}} \geqslant \frac{n_3 - \frac{1}{2} - m_3}{\sqrt{m_3}},$$

and

$$\frac{n_1 - \frac{1}{2} - m_1}{\sqrt{m_1}} \leqslant \frac{n_3 + \frac{1}{2} - m_3}{\sqrt{m_3}}.$$

This is clearly the same as the requirement that n_1, n_2, and n_3 be within 1/2 unit of the numbers that satisfy

$$\frac{n_1 - m_1}{\sqrt{m_1}} = \frac{n_2 - m_2}{\sqrt{m_2}} = \frac{n_3 - m_3}{\sqrt{m_3}}.$$

$$n_1 + n_2 + n_3 = N.$$

Relation (26) follows at once.

The conditions for the validity of (26) are not known precisely; (26) is quite accurate for values of N which are as large as $3m_r$, where $m_r = \max_\nu m_\nu$ is the largest of the m_ν. Relation (26) should not be used for values of N so small that a negative value is imputed to n_i. The condition which corresponds to (25) is

$$N \geq \sum m_\nu - \sqrt{m_1} \sum \sqrt{m_\nu}, \tag{27}$$

where m_1 is again the minimum of the mean demands: $m_1 = \min_\nu m_\nu$. In conclusion, the relation

$$n_i = \max\left(0,\ m_i + \frac{\sqrt{m_i}}{\sum \sqrt{m_i}}\left\{N - \sum m_\nu\right\}\right)$$

is a useful formula for the number of units to be allocated to the ith depot for arbitrary penalty functions (stock-out costs).

If the individual allocations add up to less than N, the excess can be allocated in any reasonable manner (say, 1 unit to each of the depots which is still empty).

12. USE OF A CENTRAL WAREHOUSE.

A distribution system sometimes consists of a central warehouse which serves a system of several satellite depots. If redistribution between depots cannot occur, stocking of items in the central warehouse for distribution to depots on demand can sometimes reduce stock-out costs. Although models constructed in this section are oversimplified, they provide useful general guidelines for establishing a stocking policy.

As a first example, consider an item in low demand, for which the economical quantity to stock in the system is one unit. (It is clear that this situation is conceivable; indeed if holding costs are sufficiently high in relation to other costs, the economical stock is no units.) The supply system is assumed to consist of a central warehouse and two depots, the latter being subject to demands ξ_1 and ξ_2 with distribution functions $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$. It is further assumed that a demand which can be met from the central warehouse but not from depot stock entails a penalty a_0, while a demand which cannot be met from either stock entails a greater penalty.

Tables 2 and 3 concern two allocations, the corresponding stock-out costs (penalties), and the differences between these respective stock-out costs.

TABLE 2

Tables of Possible Demands and Corresponding Costs

Allocation #1 (Stocks shown in boxes) Allocation #2

[1] ..... Central warehouse ..... [0]

[0] [0] ..... Satellite depots ..... [1] [0]

Costs

Depot #2

Depot #1 \ ξ_1 / ξ_2	0	1	2	3
0	0	a_0	a_0+a_1	a_0+a_2
1	a_0	a_0+a_1	a_0+a_2	a_0+a_3
2	a_0+a_1	a_0+a_2	a_0+a_3	
3	a_0+a_2	a_0+a_3		

(i) a single unit stocked centrally.

Depot #2

Depot #1 \ ξ_1 / ξ_2	0	1	2	3
0	0	0	a_1	a_2
1	a_1	a_1	a_2	a_3
2	a_2	a_2	a_3	
3	a_3	a_3		

(ii) a single unit stocked in a satellite depot.

TABLE 3

Table of Differences of Entries in Table 2

ξ_2 \ ξ_1	0	1	2	3
0	0	$-a_0$	$-a_0$	$-a_0$
1	a_1-a_0	$-a_0$	$-a_0$	$-a_0$
2	$a_2-a_1-a_0$	$-a_0$	$-a_0$	
3	$a_3-a_2-a_0$	$-a_0$		

The quantities $a_0, a_1, a_2, \ldots$, are all positive, and are subject to the restrictions

$$0 < a_0 < a_1 \leqq a_2 \leqq \ldots;$$

$$a_0 \ll a_1.$$

The last table shows that the expected value of the difference is

$$Z = a_1\phi_1(0)\phi_2(1) + (a_2 - a_1)\phi_1(0)\phi_2(2) + (a_3 - a_2)\phi_1(0)\phi_2(3) + \ldots - a_0[1 - \phi_1(0)\phi_2(0)].$$

First suppose that the system stock-out cost is independent of the number of units of shortage. Then $a_1 = a_2 = a_3 = \ldots$, and the difference in expected costs has the value

$$Z = a_0\phi_1(0)\phi_2(1)\left[\frac{a_1}{a_0} - \frac{1 - \phi_1(0)\phi_2(0)}{\phi_1(0)\phi_2(1)}\right].$$

This quantity is positive if the relation

$$\frac{a_1}{a_0} > \frac{1 - \phi_1(0)\phi_2(0)}{\phi_1(0)\phi_2(1)} \tag{28}$$

holds.*

Another interpretation of the meaning of the quantity Z can be given. Suppose arrangement #2 is modified to permit depot 2 to have access to the item stored in depot 1, subject to payment of penalties in the form of transportation and order costs. The quantity Z measures, in a certain sense, the amount of saving that must inure to the system before this accessibility can result in economic advantage. The details of the analysis of this interpretation are given below.

The value of the ratio

$$\frac{1 - \phi_1(0)\phi_2(0)}{\phi_1(0)\phi_2(1)}$$

depends on the demand functions $\phi_1(\xi)$ and $\phi_2(\xi)$. If the demand functions $\phi_i(\xi_i)$ are of Poisson type, with means m_i, the ratio in question has the value

$$\frac{e^{m_1+m_2} - 1}{m_1 + m_2} \cdot \frac{m_1 + m_2}{m_2}. \tag{29}$$

If m_1 and m_2 are sufficiently small, this ratio is close to $1 + (m_1/m_2)$.

As an example, if m_1 and m_2 are both about 1/2, the single item should be stored in a central warehouse if the cost of a system shortage exceeds three times [i.e., $(e-1)(1+1)$ times] the cost of transporting the single item to the depot at which the demand first occurs. If, however, the means m_1 and m_2 are far from equal but are both small, it may be more advantageous to stock the single item at the depot which is known to experience the greater average demand. Condition (28) is the one that governs the strategy.

*If $\phi_1(0)\phi_2(1) = 0$, the quantity Z is never positive; however, the condition $\phi_1(0) = 0$ cannot hold in a genuine problem, because if it did, the deterministic demand for a positive number of items could be met by stocking them.

When there are $r > 2$ depots, each of which experiences a low demand, the condition (28) must be replaced by the condition

$$\frac{a_1}{a_0} > \frac{1-\phi_1(0)\phi_2(0) \ . \ . \ . \ \phi_r(0)}{\phi_1(0) \sum_{t=2}^{r} \phi_t(1) \prod_{l\neq t,\, l=2}^{r} \phi_l(0)} \tag{30}$$

in which the denominator is the sum of $r-1$ terms, each of which is a product

$$\phi_1(0) \ . \ . \ . \ \phi_{t-1}(0)\phi_t(1)\phi_{t+1}(0) \ . \ . \ . \ \phi_r(0)$$

of demand functions, the t-th ($t=2, \ . \ . \ ., r$) of which is evaluated at 1, the others being evaluated at 0. For the case of Poisson demand, the ratio (29) generalizes to

$$\frac{e^m-1}{m}\left(1+\frac{m_1}{m-m_1}\right), \tag{31}$$

where

$$m = m_1 + m_2 + \ . \ . \ . \ + m_r.$$

This shows that if the demands are approximately equal, it is usually better to stock a central warehouse (since the relation $a_1/a_0 > 2$ usually holds). This also shows that if the demand at one depot has a mean which greatly exceeds the mean total demand at the other depots, it is better (as is intuitively clear) to stock the item at the depot which experiences the disproportionate share of the demand. The disproportion in mean demands needed to bring forth this conclusion is given by the relation.

$$\frac{a_1}{a_0} < \frac{e^m-1}{m} + \left(1+\frac{m_1}{m-m_1}\right);$$

for example if a_1/a_0 is 60 and m is about 1, it is economical to stock the single item at one depot that accounts for 97 percent of the demand, since

$$1-\frac{1.7}{a_1/a_0} \approx 0.97.$$

Even when a_1/a_0 has the very low value 10, use of a central warehouse is uneconomical only if one depot accounts for 83 percent or more of the demand.

Now suppose that an item is in low demand, but that a central warehouse is not used; suppose further that the item, even though stored in one of several depots at the same echelon level, can be made available to sister depots if necessary. An analysis similar to that carried out above shows that if the demand is Poisson, the single item should be stored in the depot (or one of the depots) that experiences the greatest mean demand, provided that the cost of redistribution from one depot to another is not greatly in excess of the expense that would have resulted if the item had been supplied from a central warehouse. In a multistockpile system it may not be easy to compare these costs in practice. Factors involved in the supply cost include receiving, packing and unpacking, locating the item when it is demanded, ascertaining its availability, ordering it, and transporting it.

If the cost of redistributing an item from the depot in which the item is stored to the depot demanding it is two or more times the cost involved in supplying the item from a central warehouse, it may be cheaper to use this latter system.

13. LINEAR PENALTY FUNCTION.

Now suppose (more realistically) that the (stock-out) penalty a_s for having to go outside the system for s items depends linearly on the number of items:*

$$a_s = \alpha + \beta S,\ \alpha > 0,\ \beta \geq 0.$$

In this case, the value of Z is

$$Z = a_0\phi_1(0)\left\{\frac{\alpha\phi_2(1)+\beta[1-\phi_2(0)]}{a_0} - \frac{1-\phi_1(0)\phi_2(0)}{\phi_1(0)}\right\}. \tag{32}$$

The condition that this be positive is somewhat different from (28) unless α/β has a large value. For Poisson demands, however, the conditions come to about the same thing, since in this case, (32) is positive if the condition

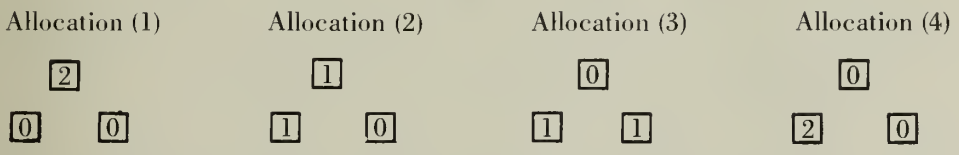
$$\frac{a_1}{a_0} + \frac{\beta}{a_0}\left\{\frac{e^{m_2}-1}{m_2} - 1\right\} > \frac{e^{m_1+m_2}-1}{m_1+m_2}\left(1+\frac{m_1}{m_2}\right) \tag{33}$$

holds, and the $\{\ \}$ in the left member is about $\frac{1}{2}m_2$ when m_2 is close to 0. Thus (33) is very close to (28) when m_2 is sufficiently close to 0.

This agrees with the intuitive argument that if m_1 and $m_2 \to 0$, the penalty is almost completely described by the values of a_0 and a_1; demands for more than two items are relatively rare.

When two units are to be allocated to a system consisting of a warehouse and two satellite depots, there are six possible allocations. An indication of the complexity of the solution to the problem of finding the best allocation can be given by showing the solution for a special case.

Suppose depots 1 and 2 experience Poisson demand with means m_1 and m_2, respectively. Suppose further that the system stock-out cost is a_1, independent of the number of units of shortage, and suppose that the cost of any shipment or shipments of 1 or 2 units from the central warehouse to one or both depots is the constant a_0.



Allocation (1) is cheaper than allocation (2) if $\dfrac{a_1}{a_0} > \dfrac{2m_1}{m_2^2}$.

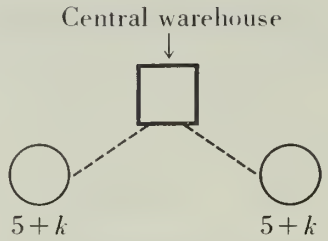
Allocation (1) is cheaper than allocation (3) if $\dfrac{a_1}{a_0} > \dfrac{e^{m_1+m_2}-1}{m_1+m_2} \cdot \dfrac{2(m_1+m_2)}{(m_1^2+m_2^2)}$.

Allocation (1) is cheaper than allocation (4) if $\dfrac{a_1}{a_0} > \dfrac{e^{m_1+m_2}-1}{m_1+m_2} \cdot \dfrac{m_1+m_2}{m_2} \cdot \dfrac{2}{2m_1+m_2+2}$.

*βs is the monetary value of the inconvenience caused by a shortage of s items.

The complexity of the answer to this oversimplified problem is not encouraging; to determine the best stocking policy, it is necessary to know the penalties and mean demands rather accurately.

For a multistockpile supply system using a central warehouse, another approach to the problem of determining the best stocking policy is simulation, i.e., a Monte Carlo computation. One such computation has been completed. It was supposed that 30 items are to be allocated among two depots and a central warehouse. The depots are subject to Poisson demand with mean values 10 and 20, respectively. Shipment *from* either depot was forbidden (transportation cost infinite). Cost of shipment of k items *to* either depot from the central warehouse was taken to be $\$5+k$ $(k>0)$; cost of a shortage of t items at a depot was taken to be $\$20+3t$ $(t>0)$. It was required to calculate the expected costs of various allocations.



$$\phi_1(\xi)=\frac{e^{-10}10^{\xi}}{\xi!} \qquad \phi_2(\xi)=\frac{e^{-20}20^{\xi}}{\xi!}$$

Representative costs are as follows.

	Allocation (1)	Allocation (2)	Allocation (3)
Central Warehouse:	9	6	3
Branches:	7 14	8 16	9 18
Shortage Penalty:	\$32	\$23	\$32

This table shows that for the penalty functions used, it is probably economical to stock about 6 of the 30 items in a central store, where they can be made available at reasonable cost to each of the depots. But the other items should be stocked at the depots themselves, to avoid the delay entailed in requisitioning the central warehouse.

ACKNOWLEDGEMENT

This article reflects the lively research atmosphere of the productive Operations Research group at Stanford Research Institute, of which I was a member when this article was roughed out. Specific help from D. A. D'Esopo improved the statement of many of the results.

The research here reported was generously supported by the Bureau of Supplies and Accounts, U.S. Navy. Responsibility for the article, however, rests solely with the author.

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OPTIMUM QUANTILES FOR THE LINEAR ESTIMATION OF THE PARAMETERS OF THE EXTREME VALUE DISTRIBUTION IN COMPLETE AND CENSORED SAMPLES*

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ABSTRACT

The present study is concerned with the determination of a few observations from a sufficiently large complete or censored sample from the extreme value distribution with location and scale parameters μ and σ, respectively, such that the asymptotically best linear unbiased estimators (ABLUE) of the parameters in Ref. [24] yield high efficiencies among other choices of the same number of observations. (All efficiencies considered are relative to the Cramér-Rao lower bounds for regular unbiased estimators.) The study is on the asymptotic theory and under Type II censoring scheme.

For the estimation of μ when σ is known, it has been proved that there exists a unique optimum spacing whether the sample is complete, right censored, left censored, or doubly censored. Several tables are prepared to aid in the numerical computation of the estimates as well as to furnish their efficiencies.

For the estimation of σ when μ is known, it has been observed that there does not exist a unique optimum spacing. Accordingly we have obtained a spacing based on a complete sample which yields high efficiency. A similar table as above is prepared.

When both μ and σ are unknown, we have considered four different spacings based on a complete sample and chosen the one yielding highest efficiency. A table of the efficiencies is also prepared.

Finally we apply the above results for the estimation of the scale and/or shape parameters of the Weibull distribution.

1. INTRODUCTION

Suppose that in a random sample of size n from a continuous distribution with distribution function $H[(x-\mu)/\sigma]$, the ordered observations are $x_{(1)} < \ldots < x_{(n)}$. The problem of estimating the parameters μ and σ simultaneously or one of them when the other is known based on k (where k is much less than n or r according as the sample is complete or Type II censored, r being the number of observations in the censored sample) sample observations, say, $x_{(n_1)} < \ldots < x_{(n_k)}$ frequently arises in practice. Linear estimators of the form

$$\sum_{i=1}^{k} c_i x_{(n_i)}$$

(two such differing only in the coefficients c_i when both μ and σ are to be estimated) are often of interest, and the problem is to obtain the set $\{n_1, \ldots, n_k\}$ of ranks n_i, such that the estimators based on the corresponding order statistics (called the 'optimum quantiles') possess minimum variance property among the $\binom{n}{k}\left(\text{or}\binom{r}{k}\right)$ possible estimators.

*This research was partially supported by the National Research Council of Canada and was completed while the authors were fellows at the 1968 Summer Research Institute of the Canadian Mathematical Congress.

Consider the random variable x having the extreme value distribution with density function

$$\frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma}-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0, \tag{1.1}$$

and distribution function

$$\exp\{-\exp[-(x-\mu)/\sigma]\},$$

where μ and σ are, respectively, the location and scale parameters. Clearly, the standardized random variable $u=(x-\mu)/\sigma$ has the density function

$$f(u)=\exp\{-u-\exp(-u)\}, \qquad -\infty < u < \infty \tag{1.2}$$

and the distribution function

$$F(u)=\exp\{-\exp(-u)\}. \tag{1.3}$$

The distribution given by (1.1) is usually called the Type I asymptotic distribution of largest values. The distribution with distribution function $1-\exp\{-\exp[(x-\mu)/\sigma]\}$ is usually called the Type I asymptotic distribution of smallest values. It may be noted that if x has the Type I asymptotic distribution of largest values, then $-x$ has that of smallest values. Further relations between these two distributions were given by Lieblein and Zelen ([13], p. 293). In the present paper the extreme value distribution refers to the Type I asymptotic distribution of largest values.

Gumbel made an extensive study of the theory of extreme values. Various results by him and others up to 1957 were summarized in his book [4]. Since then point estimation of μ and/or σ has been considered by a number of authors. A good survey of results up to 1967 was given by Harter and Moore [6] and Mann [20]. Mann has also made extensive studies on extreme value distributions; some of her results can be found in References [14] to [20]. A comparison of estimators proposed in recent years is provided in [2] and [20]. Thus we do not attempt a review of published work in this paper; however, mention will be made of some of the articles in appropriate sections when we compare our estimators with others.

It is well known that the transformation $\exp(x)$ has the two-parameter Weibull distribution. So results on the extreme value distribution may be applied to the Weibull distribution. Section 5 indicates such applications.

Since, in this paper, we are only interested in large samples estimation, no reference will be made of some of the articles concerning the small samples situation.

Let the ordered observations in a random sample of size n from the distribution (1.1) be $x_{(1)} < \ldots < x_{(n)}$. Suppose $\lambda_1, \ldots, \lambda_k$ are $k(<n)$ fixed real numbers such that $0<\lambda_1<\ldots<\lambda_k<1$. Define the integers $n_1, \ldots, n_k$ by the relations $n_i=[n\lambda_i]+1$, $i=1, \ldots, k$, where $[n\lambda_i]$ denotes the greatest integer not exceeding $n\lambda_i$. Consider the k sample quantiles $x_{(n_1)}, \ldots, x_{(n_k)}$ of orders $\lambda_1, \ldots, \lambda_k$, respectively, where the ranks $n_i (i=1, \ldots, k)$ satisfy the inequalities $1 \leq n_1 < \ldots < n_k \leq n$. The values of $\lambda_1, \ldots, \lambda_k$ for which the relative efficiency of an estimator attains its maximum value will be called an 'optimum spacing.' Throughout the paper we assume n to be sufficiently large.

Let u_i denote the λ_i-quantile of the standardized population, that is, $F(u_i)=\lambda_i (i=1, \ldots, k)$. According to Mosteller [23], the joint asymptotic $(n\to\infty)$ distribution of the k sample quantiles

$x_{(n_1)}, \ldots, x_{(n_k)}$ is a k-variate normal distribution with means $\mu+\sigma u_i$, $i=1, \ldots, k$, and dispersion matrix $W=(W_{ij})$ with elements

$$W_{ij}=W_{ji}=\frac{\lambda_i(1-\lambda_j)}{nf(u_i)f(u_j)} \quad i \leq j;\ i, j=1, \ldots, k.$$

The inverse of W denoted by $M=(M_{ij})$, has the elements

$$M_{ii}=\frac{\lambda_{i+1}-\lambda_{i-1}}{(\lambda_{i+1}-\lambda_i)(\lambda_i-\lambda_{i-1})}\, n\{f(u_i)\}^2,$$

$$M_{i-1,i}=M_{i,i-1}=-\frac{nf(u_i)f(u_{i-1})}{\lambda_i-\lambda_{i-1}},$$

and

$$M_{ij}=0 \text{ for } |i-j|>1,$$

where

$$u_i=-\ln[\ln(1/\lambda_i)], \quad i=1, \ldots, k.$$

By the generalized Gauss-Markov theorem [11], the asymptotically best linear unbiased estimators (ABLUE) of μ and σ based on k sample quantiles [24] are as follows.

CASE 1. σ is known. The ABLUE of μ is

$$\mu^*=\sum_{i=1}^{k} a_i x_{(n_i)}-\sigma K_3/K_1 \tag{1.4}$$

with variance

$$V(\mu^*)=\frac{\sigma^2}{n}\frac{1}{K_1}. \tag{1.5}$$

CASE 2. μ is known. The ABLUE of σ is

$$\sigma^*=\sum_{i=1}^{k} b_i x_{(n_i)}-\mu K_3/K_2 \tag{1.6}$$

with variance

$$V(\sigma^*)=\frac{\sigma^2}{n}\frac{1}{K_2}. \tag{1.7}$$

CASE 3. Both μ and σ are unknown. The ABLUE of μ and σ are

$$\hat{\mu}=\sum_{i=1}^{k} a_i' x_{(n_i)}, \qquad \hat{\sigma}=\sum_{i=1}^{k} b_i' x_{(n_i)} \tag{1.8}$$

with variances, covariance, and the generalized variance Λ given by

$$V(\hat{\mu})=\frac{\sigma^2}{n}\frac{K_2}{\Delta}, \qquad V(\hat{\sigma})=\frac{\sigma^2}{n}\frac{K_1}{\Delta}, \qquad \text{Cov}(\hat{\mu}, \hat{\sigma})=-\frac{\sigma^2}{n}\frac{K_3}{\Delta}, \qquad \Lambda=\frac{\sigma^4}{n^2}\frac{1}{\Delta}. \tag{1.9}$$

The undefined quantities above have the following meanings:

$$K_{1i}=[(f_i-f_{i-1})/(\lambda_i-\lambda_{i-1})-(f_{i+1}-f_i)/(\lambda_{i+1}-\lambda_i)]f_i,$$

$$K_{2i}=[(u_if_i-u_{i-1}f_{i-1})/(\lambda_i-\lambda_{i-1})-(u_{i+1}f_{i+1}-u_if_i)/(\lambda_{i+1}-\lambda_i)]f_i,$$

$$K_1=\sum_{i=1}^{k}K_{1i},\ K_2=\sum_{i=1}^{k}u_iK_{2i},\ K_3=\sum_{i=1}^{k}K_{2i},\ \Delta=K_1K_2-K_3^2,$$

$$a_i=\sum_{i=1}^{k}K_{1i}/K_1,\ b_i=\sum_{i=1}^{k}K_{2i}/K_2,$$

$$a_i'=(K_2K_{1i}-K_3K_{2i})/\Delta,\ b_i'=(K_1K_{2i}-K_3K_{1i})/\Delta,$$

and we set $f_i=f(u_i)\ (i=0, 1, \ldots, k+1)$, $\lambda_0=0$, $\lambda_{k+1}=1$, $f_0=f_{k+1}=u_0f_0=u_{k+1}f_{k+1}=0$.

The following notations shall be used throughout:

$$\bar{S}(\alpha, \beta)=\{(\lambda_1, \ldots, \lambda_k)\,|\,\alpha\leq\lambda_1\leq\ldots\leq\lambda_k\leq\beta\},\ \text{where } 0\leq\alpha\leq\beta\leq 1;$$

$$S(\alpha, \beta)=\{(\lambda_1, \ldots, \lambda_k)\,|\,\alpha\leq\lambda_1<\ldots<\lambda_k\leq\beta\},\ \text{where } \alpha<\lambda_1 \text{ if } \alpha=0 \text{ and } \lambda_k<\beta \text{ if } \beta=1;$$

$$S(\alpha, \beta]=\{(\lambda_1, \ldots, \lambda_k)\,|\,\alpha\leq\lambda_1<\ldots<\lambda_k=\beta\} \text{ where } \beta<1, \text{ and } \alpha<\lambda_1 \text{ if } \alpha=0;$$

$$S[\alpha, \beta)=\{(\lambda_1, \ldots, \lambda_k)\,|\,\alpha=\lambda_1<\ldots<\lambda_k\leq\beta\} \text{ where } \alpha>0, \text{ and } \lambda_k<\beta \text{ if } \beta=1;$$

$$S[\alpha, \beta]=\{(\lambda_1, \ldots, \lambda_k)\,|\,\alpha=\lambda_1<\ldots<\lambda_k=\beta\} \text{ where } \alpha>0 \text{ and } \beta<1.$$

As an illustration, $S(0, \beta]=\{(\lambda_1<\ldots<\lambda_k)\,|\,0<\lambda_1<\ldots<\lambda_k=\beta\}$ where $\beta<1$.

2. OPTIMUM QUANTILES FOR THE ESTIMATION OF μ WHEN σ IS KNOWN

2.1. Complete Sample

In order to determine the k optimum quantiles for the estimation of μ, we have to minimize $V(\mu^*)$ in (1.5) with respect to $u_1, \ldots, u_k$ over the domain $-\infty\leq u_1\leq\ldots\leq u_k\leq\infty$. Equivalently, we maximize K_1 which can be written in the alternative form

$$K_1=K_1(u_1, \ldots, u_k)=\sum_{i=1}^{k+1}\frac{(f_i-f_{i+1})^2}{\lambda_i-\lambda_{i-1}}. \tag{2.1}$$

Clearly, we may write $f_i=f(u_i)=-\lambda_i \ln\lambda_i=-g(\lambda_i)=-g_i$, say. Making this transformation in (2.1) we have

$$K_1=K_1(\lambda_1, \ldots, \lambda_k)=\sum_{i=1}^{k+1}\frac{(g_i-g_{i-1})^2}{\lambda_i-\lambda_{i-1}} \tag{2.2}$$

defined on $\bar{S}(0, 1)$.

We shall need the following lemmas in the sequel.

LEMMA 2.1: K_1 in (2.2) defined on $\bar{S}(0, 1)$ attains its maximum at an interior point of $\bar{S}(0, 1)$.

PROOF: Since for $i=1, \ldots, k+1$, $(g_i-g_{i-1})^2/(\lambda_i-\lambda_{i-1})\to 0$ as $|\lambda_i-\lambda_{i-1}|\to 0$, K_1 is continuous on the compact set $\bar{S}(0, 1)$. So K_1 attains a maximum at some point, say, $(\lambda_1, \ldots, \lambda_k)$ in

$\bar{S}(0, 1)$. We have to show that $\lambda_{i-1} \neq \lambda_i$ for $1 \leq i \leq k+1$. Since $d^2g(y)/dy^2 = 1/y > 0$ for $y \in [0, 1]$, $g(y)$ is concave, and so for any $0 \leq y_1 < y_2 < y_3 \leq 1$, we have

$$\frac{g(y_3)-g(y_2)}{y_3-y_2} \neq \frac{g(y_2)-g(y_1)}{y_2-y_1}. \tag{2.3}$$

Now, suppose $\lambda_{i-1} = \lambda_i < \lambda_{i+1}$ for some $i \in \{1, \ldots, k\}$. By (2.3), for any λ such that $\lambda_{i-1} < \lambda < \lambda_{i+1}$ the difference

$$K_1(\lambda_1, \ldots, \lambda_{i-1}, \lambda, \lambda_{i+1}, \ldots, \lambda_k) - K_1(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_k)$$

$$= \frac{(g-g_{i-1})^2}{\lambda-\lambda_i} + \frac{(g_{i+1}-g)^2}{\lambda_{i+1}-\lambda} - \frac{(g_{i+1}-g_{i-1})^2}{\lambda_{i+1}-\lambda_{i-1}} = ab(\lambda_{i+1}-\lambda_{i-1})(A-B)^2 > 0$$

where $a = (\lambda-\lambda_{i-1})/(\lambda_{i+1}-\lambda_{i-1}) > 0$, $b = (\lambda_{i+1}-\lambda)/(\lambda_{i+1}-\lambda_i) > 0$, $A = (g-g_{i-1})/(\lambda-\lambda_{i-1})$ and $B = (g_{i+1}-g)/(\lambda_{i+1}-\lambda)$. The contradiction establishes the lemma.

Thus in order to investigate the maximum of K_1 in (2.2), it is enough to consider the set $S(0, 1)$.

LEMMA 2.2: A point which maximizes K_1 in (2.2) defined on $S(0, 1)$ must satisfy the system of equations

$$\frac{g_{i+1}-g_i}{\lambda_{i+1}-\lambda_i} + \frac{g_i-g_{i-1}}{\lambda_i-\lambda_{i-1}} - 2g_i' = 0 \qquad (i = 1, \ldots, k) \tag{2.4}$$

where $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k < \lambda_{k+1} = 1$ and $g_i' = g'(\lambda_i) = dg(\lambda_i)/d\lambda_i$.

PROOF: Since K_1 attains its maximum at a point in $S(0, 1)$ which is an interior point, this point must satisfy the system of equations

$$\frac{\partial K_1}{\partial \lambda_i} = \left[\frac{g_{i+1}-g_i}{\lambda_{i+1}-\lambda_i} - \frac{g_i-g_{i-1}}{\lambda_i-\lambda_{i-1}}\right]\left[\frac{g_{i+1}-g_i}{\lambda_{i+1}-\lambda_i} + \frac{g_i-g_{i-1}}{\lambda_i-\lambda_{i-1}} - 2g_i'\right] = 0 \qquad (i = 1, \ldots, k). \tag{2.5}$$

But the first factor on the right-hand side of (2.5) is zero if and only if $\lambda_{i-1} = \lambda_i = \lambda_{i+1}$. Hence the lemma is proved.

LEMMA 2.3: Let $h_1(t) = -[\ln t/(1-t)]/2$ and $h_2(t) = 1 - th_1(t)$, $0 \leq a \leq t \leq b \leq 1$. Then for any fixed $t' \in [a, b]$, there exists a unique $t'' \in [t', 1]$ such that $h_1(t'') = h_2(t')$.

PROOF: Since $h_1(t)$ and $h_2(t)$ are strictly decreasing on $[0, 1]$, $h_1(t) > h_2(t)$ on $[0, 1]$, and $h_1(0) = \infty$, $h_2(0) = 1$, $h_1(1) = h_2(1) = 1/2$, the assertion can easily be verified through Figure 1.

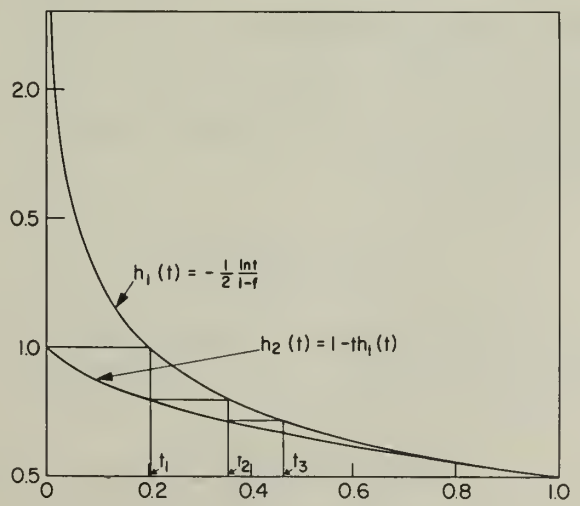

FIGURE 1. The graphs of $h_1(t)$ and $h_2(t)$ as functions of t

THEOREM 2.1: The system of Eqs. (2.4) has a unique solution $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ in $S(0, 1)$ which corresponds to the maximum of K_1 in (2.2) defined on $S(0, 1)$.

PROOF: Let $t_i = \lambda_i/\lambda_{i+1}$ $(i = 0, 1, \ldots, k)$. Clearly $0 \leq t_i < 1$ and $(g_{i+1} - g_i)/(\lambda_{i+1} - \lambda_i) = [(\lambda_{i+1} - \lambda_i) \ln \lambda_i - \lambda_{i+1} \ln (\lambda_i/\lambda_{i+1})]/[\lambda_{i+1} - \lambda_i] = \ln \lambda_i - \ln t_i/(1 - t_i)$. Then (2.4) reduces to

$$\frac{\ln t_i}{1 - t_i} + \frac{t_{i-1} \ln t_{i-1}}{1 - t_{i-1}} + 2 = 0 \qquad (i = 1, \ldots, k) \tag{2.6}$$

which can be written as

$$h_1(t_i) = 1 - t_{i-1} h_1(t_{i-1}) = h_2(t_{i-1}) \qquad (i = 1, \ldots, k)$$

where the functions $h_1(t)$ and $h_2(t)$ are as in Lemma 2.3. Since $t_0 = 0$, by LEMMA 2.3, t_1 is uniquely determined by the equation $h_1(t_1) = h_2(t_0) = 1$. Since t_1 is now known, t_2 is uniquely determined by the equation $h_1(t_2) = h_2(t_1)$. Continuing this process, we can obtain for any k, the unique set of numbers denoted by $\{t_1^\circ, \ldots, t_k^\circ\}$.

Now, since $h_1(t) \geq h_2(t)$ for $t \in [0, 1]$ with equality holding if, and only if, $t = 1$, we must have, for every finite k,

$$0 < t_1^\circ < t_2^\circ < \ldots < t_k^\circ < 1. \tag{2.7}$$

The relation $t_k^\circ = \lambda_k/\lambda_{k+1}$ with $\lambda_{k+1} = 1$ thus determines $\lambda_k = \lambda_k^\circ (= t_k^\circ)$ uniquely. By (2.7) and the relation $\lambda_i = t_i \lambda_{i+1} (i = 1, \ldots, k)$, we thus obtain a unique point $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ satisfying the inequalities $0 < \lambda_1^\circ < \ldots < \lambda_k^\circ < 1$. This point $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ is easily seen from (2.6) to be the unique solution of (2.4). That this solution corresponds to the maximum of K_1 in (2.2) defined on $S(0, 1)$ follows from Lemmas 2.1 and 2.2. Thus the theorem is proved.

It is of interest to note that we can calculate t_i° up to as large value of i as we wish. For any fixed k, the set $\{t_1^\circ, \ldots, t_k^\circ\}$ is the relevant one. Table 1 furnishes the values of t_i° from $i = 0(1)49$.

We observe from the proof of Lemma 2.1 that the value of K_1 in (2.2) increases with the number of λ_i's. So, as $\max |\lambda_i - \lambda_{i-1}|$ tends to zero,

$$\lim K_1(\lambda_1, \ldots, \lambda_k) = \lim \sum_{i=1}^{k+1} \left(\frac{g_i - g_{i-1}}{\lambda_i - \lambda_{i-1}} \right)^2 (\lambda_i - \lambda_{i-1}) = \int_0^1 \left(\frac{dg}{d\lambda} \right)^2 d\lambda = \int_0^1 \left(\frac{f'}{f} \right)^2 dF = 1$$

where f and F are given by (1.2) and (1.3), respectively, and $f' = df/du$. Thus the amount of information of the original sample with respect to μ is

$$I_0(\mu) = \frac{n}{\sigma^2} E\left(\frac{f'}{f}\right)^2 = \frac{n}{\sigma^2}.$$

So the Cramér-Rao lower bound for regular unbiased estimators of μ is σ^2/n. We now define the asymptotic relative efficiency (ARE) of the estimator μ^* based on complete or censored sample as

$$ARE(\mu^*) = \frac{1/I_0(\mu)}{V(\mu^*)} = K_1,$$

where $V(\mu^*)$ is given by (1.5).

The optimum spacings $\lambda_1^\circ, \ldots, \lambda_k^\circ$ with the corresponding coefficients $a_1, \ldots, a_k$, and the values of K_1 and K_3/K_1 were computed for $k=1(1)20$. For $k=1(1)15$, these values coincide with those computed by Hassanein (Tables 1 and 3 of [8]); however, he did not prove the uniqueness of the λ_i°.

TABLE 1

Table furnishing the values of t_i° from $i=0(1)49$

	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.2031	0.3614	0.4704	0.5486	0.6069	0.6521	0.6880	0.7173	0.7415
1	0.7619	0.7794	0.7945	0.8076	0.8192	0.8295	0.8386	0.8468	0.8543	0.8610
2	0.8672	0.8728	0.8780	0.8827	0.8871	0.8912	0.8950	0.8986	0.9019	0.9050
3	0.9079	0.9107	0.9132	0.9157	0.9180	0.9202	0.9222	0.9242	0.9261	0.9278
4	0.9295	0.9312	0.9327	0.9342	0.9356	0.9369	0.9382	0.9395	0.9407	0.9418

Although the problem of small samples estimation of the parameters of the extreme value distribution based on censored samples has received attention ([5, 7, 9, 15, 16, 18, 19, 21, 27, 28]), the corresponding large samples problem has hardly been studied. In subsections 2.2, 2.3, and 2.4 we show that, for large samples estimation of μ (when σ is known) based on singly or doubly censored samples, there exists a unique optimum spacing.

2.2. Right Censored Sample

Let $x_{(1)} < \ldots < x_{(n)}$ be an ordered sample of size n and $x_{(1)} < \ldots < x_{(r_2-1)} < x_{(r_2)}$ be the uncensored portion of size r_2 in which $x_{(r_2)}$ is the largest available observation. The integer r_2 is the rank of the observation $x_{(r_2)}$ in the complete sample and is determined by a prefixed real number β such that $0<\beta<1$, using the relation $r_2=[n\beta]+1$. Then $(1-\beta)$ is the proportion of censoring on the right.

For a given $k(<r_2)$, consider the sample quantiles $x_{(n_1)}, \ldots, x_{(n_k)}$ with ranks $n_1, \ldots, n_k$ determined by k fixed real numbers $\lambda_1, \ldots, \lambda_k$, such that $0<\lambda_1<\ldots<\lambda_k\leq\beta$ and $n_i=[n\lambda_i]+1$, $i=1, \ldots, k$. Define $\lambda_0=0$ and $\lambda_{k+1}=1$.

Based on k sample quantiles $x_{(n_1)}, \ldots, x_{(n_k)}$, the ABLUE of μ when σ is known and its variance are given by (1.4) and (1.5), respectively.

In order to determine the k optimum quantiles for the estimation of μ, we have to minimize $V(\mu^*)$ in (1.5) with respect to $u_1, \ldots, u_k$ over the domain $-\infty\leq u_1\leq\ldots\leq u_k\leq -\ln\ln(1/\beta)$. Equivalently, we maximize K_1 in (2.2) on $\bar{S}(0, \beta)$. By Theorem 2.1, K_1 has a unique maximum at an interior point $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ of $\bar{S}(0, 1)$. For the maximum of K_1 on $\bar{S}(0, \beta)$, we have to consider the following two cases: (i) $\lambda_k^\circ\leq\beta$ and (ii) $\lambda_k^\circ>\beta$.

In the first case, we take the maximum of K_1 corresponding to $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ and the k optimum quantiles are the same as those in the complete sample.

For the second case, we prove the following theorem.

THEOREM 2.2: (i) The function K_1 in (2.2) defined on $\bar{S}(0, \beta)$ with $\lambda_k\leq\beta<\lambda_k^\circ$ attains its maximum at a point $(\lambda_1', \ldots, \lambda_k')$ in $\bar{S}(0, \beta)$.

(ii) $(\lambda_1', \ldots, \lambda_k')$ is the unique solution of the system of equations

$$\frac{g_{i+1}-g_i}{\lambda_{i+1}-\lambda_i}+\frac{g_i-g_{i-1}}{\lambda_i-\lambda_{i-1}}-2g_i'=0 \qquad (i=1, \ldots, k-1) \tag{2.9}$$

with $0=\lambda_0<\lambda_1<\ldots<\lambda_k=\beta$.

PROOF: (i) By Theorem 2.1, the function K_1 defined on $\bar{S}(0, 1)$ has a unique maximum at an interior point $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ of $\bar{S}(0, 1)$. We now show that if $\beta < \lambda_k^\circ$, then K_1 cannot have a maximum at an interior point of $\bar{S}(0, \beta)$.

Assume that K_1 attains its maximum at an interior point $(\lambda_1^*, \ldots, \lambda_k^*)$ of $\bar{S}(0, \beta)$. Then this point gives K_1 on $\bar{S}(0, 1)$ a relative maximum and hence must satisfy (2.4). But by Theorem 2.1, (2.4) has a unique solution $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$. So $\beta < \lambda_k^\circ$ implies that (2.4) has two distinct solutions. This contradiction establishes the above assertion.

Therefore, the maximum of K_1 on $\bar{S}(0, \beta)$ occurs on the boundary. Proceeding as in Lemma 2.1, we see that $\lambda_{i-1} \neq \lambda_i$ for $1 \leq i \leq k$. So we must have $\lambda_k' = \beta$. This proves the first part of the theorem.

(ii) From (i) we observe that the maximization of K_1 on $\bar{S}(0, \beta)$ is now equivalent to that of

$$\sum_{i=1}^{k} (g_i - g_{i-1})^2 / (\lambda_2 - \lambda_{i-1_5}$$

on $\bar{S}(0, \beta)$ with respect to $\lambda_1, \ldots, \lambda_{k-1}$. Following an argument analogous to the proof of Lemma 2.2, we see that $(\lambda_1', \ldots, \lambda_k')$ is a solution of (2.9).

To prove the uniqueness of solution of (2.9), let $t_i = \lambda_i/\lambda_{i+1}$ $(i = 0, 1, \ldots, k-1)$ with $\lambda_k = \beta$. Clearly $0 \leq t_i < 1$ and $(g_{i+1} - g_i)/(\lambda_{i+1} - \lambda_i) = \ln \lambda_i - \ln t_i/(1 - t_i)$. Thus (2.9) reduces to $\ln t_i/(1 - t_i) + t_{i-1} \ln t_{i-1}/(1 - t_{i-1}) + 2 = 0$ which can be written as $h_1(t_i) = h_2(t_{i-1})$, $i = 1, \ldots, k-1$, where $h_1(t)$ and $h_2(t)$ are as in Lemma 2.3.

Procceding along similar lines as in the proof of Theorem 2.1, we obtain the unique set $\{t_0', \ldots, t_{k-1}'\}$ such that $0 = t_0' < t_1' < \ldots < t_{k-1}' < 1$. Since $t_0' = 0$, we have $t_i' = t_i^\circ$ $(i = 1, \ldots, k-1)$ where the t_i°'s are the quantities in Theorem 2.1. We can now determine λ_{k-1}' uniquely by the equation $t_{k-1}' = \lambda_{k-1}/\lambda_k = \lambda_{k-1}/\beta$, and then proceeding successively obtain the unique point $(\lambda_1', \ldots, \lambda_k')$ with $\lambda_k' = \beta$ such that $0 < \lambda_1' < \ldots < \lambda_k' = \beta$. This point is easily seen to be the unique solution of (2.9). This completes the proof of the theorem.

REMARK 1: It may be observed from the proof of Theorem 2.2 (ii) that if $(\lambda_1^{(1)}, \ldots, \lambda_{k-1}^{(1)}, \beta^{(1)})$ and $(\lambda_1^{(2)}, \ldots, \lambda_{k-1}^{(2)}, \beta^{(2)})$ are the optimum spacings for K_1 with right censoring at $\beta^{(1)}$ and $\beta^{(2)}$, respectively, and $\beta^{(1)} < \beta^{(2)} \leq \lambda_k^\circ$, then $\lambda_i^{(1)} < \lambda_i^{(2)}$ $(i = 1, \ldots, k-1)$. For, let us consider the transformation

$$t_i^\circ = \lambda_i^{(1)}/\lambda_{i+1}^{(1)} = \lambda_i^{(2)}/\lambda^{(2)}{}_{,+1} \qquad (i = 0, 1, \ldots, k-1) \tag{2.10}$$

with $\lambda_0^{(1)} = \lambda_0^{(2)} = 0$, $\lambda_k^{(1)} = \beta^{(1)}$ and $\lambda_k^{(2)} = \beta^{(2)}$ as is done in the proof of Theorem 2.2 (ii). Then from (2.10), $\beta^{(1)} < \beta^{(2)}$ implies $\lambda_i^{(1)} < \lambda_i^{(2)}$ $(i = 1, \ldots, k-1)$. In particular, if $\beta^{(2)} = \lambda_k^\circ$, then $\lambda_i^{(1)} < \lambda_i^\circ$ $(i = 1, \ldots, k-1)$. Further, it is obvious from the relation $\lambda^{(1)} = t_1^\circ \ldots t_{k-1}^\circ \beta^{(1)}$ that $\lambda^{(1)}$ decreases to zero with $\beta^{(1)}$.

REMARK 2: Alternatively, if we know that $\lambda_1^{(1)} < \lambda_1^{(2)}$, $\beta^{(1)} \leq \lambda_k^\circ$, $\beta^{(2)} \leq \lambda_k^\circ$, then (2.10) implies $\lambda_i^{(1)} < \lambda_i^{(2)} (i = 2, \ldots, k)$.

According to the definition (2.8) the *ARE* of the estimator μ^* is K_1.

The optimum spacing $\lambda_1', \ldots, \lambda_k'$ with the corresponding coefficients $a_1, \ldots, a_k$, and the values of K_1 and K_3/K_1 for $k = 2(1)5$ and various proportions of right censoring are presented in Table 2.

EXAMPLE: Let $k = 4$, $n = 90$, $\beta = 0.45$. We observe from Table 1 of [8] that this situation belongs to case (ii). From Table 2 with $k = 4$ and $\beta = 0.45$, it is easily seen that the optimum quantiles are $x_{(2)}$, $x_{(7)}$, $x_{(20)}$, $x_{(41)}$. The *ABLUE* of μ is

$$\mu^* = 0.1403x_{(2)} + 0.2531x_{(7)} + 0.3021x_{(20)} + 0.3045x_{(41)} + 0.5034\sigma$$

with $ARE(\mu^*) = 92.27\%$.

TABLE 2

Optimum spacings $\lambda_1', \ldots, \lambda_k'$ with the corresponding coefficients and efficiencies for estimating μ based on $k=2(1)5$ sample quantiles from a right censored sample for various proportions of censoring $(1-\beta)$

k	β	λ_1	λ_2	λ_3	λ_4	λ_5	a_1	a_2	a_3	a_4	a_5	K_1	K_3/K_1
2	0.20	0.0406	0.2000				0.3350	0.6650				0.7771	−0.7064
	0.23	0.0467	0.2300				0.3605	0.6395				0.7941	−0.6498
	0.25	0.0508	0.2500				0.3772	0.6228				0.8025	−0.6153
	0.28	0.0569	0.2800				0.4019	0.5981				0.8115	−0.5676
	0.30	0.0610	0.3000				0.4182	0.5818				0.8155	−0.5382
	0.33	0.0671	0.3300				0.4424	0.5576				0.8191	−0.4973
	0.35	0.0711	0.3500				0.4584	0.5416				0.8201	−0.4720
3	0.30	0.0220	0.1084	0.3000			0.1938	0.3298	0.4764			0.8673	−0.6112
	0.33	0.0242	0.1193	0.3300			0.2058	0.3437	0.4505			0.8761	−0.5761
	0.35	0.0257	0.1265	0.3500			0.2138	0.3527	0.4336			0.8805	−0.5546
	0.38	0.0279	0.1374	0.3800			0.2256	0.3656	0.4088			0.8855	−0.5249
	0.40	0.0294	0.1446	0.4000			0.2335	0.3739	0.3926			0.8878	−0.5066
	0.43	0.0316	0.1554	0.4300			0.2452	0.3860	0.3688			0.8901	−0.4813
	0.45	0.0331	0.1627	0.4500			0.2530	0.3937	0.3533			0.8908	−0.4658
4	0.40	0.0318	0.0680	0.1882	0.4000		0.1292	0.2369	0.2909	0.3429		0.9161	−0.5414
	0.43	0.0149	0.0731	0.2023	0.4300		0.1359	0.2467	0.2978	0.3196		0.9205	−0.5179
	0.45	0.0155	0.0765	0.2117	0.4500		0.1403	0.2531	0.3021	0.3045		0.9227	−0.5034
	0.48	0.0166	0.0816	0.2258	0.4800		0.1470	0.2625	0.3080	0.2824		0.9250	−0.4836
	0.50	0.0173	0.0850	0.2352	0.5000		0.1514	0.2687	0.3117	0.2681		0.9260	−0.4715
	0.53	0.0183	0.0901	0.2493	0.5300		0.1581	0.2779	0.3168	0.2472		0.9268	−0.4549
5	0.40	0.0076	0.0373	0.1032	0.2194	0.4000	0.0796	0.1566	0.2136	0.2361	0.3141	0.9305	−0.5586
	0.43	0.0082	0.0401	0.1110	0.2359	0.4300	0.0838	0.1637	0.2210	0.2403	0.2912	0.9359	−0.5358
	0.45	0.0085	0.0420	0.1161	0.2469	0.4500	0.0866	0.1683	0.2258	0.2428	0.2764	0.9388	−0.5219
	0.48	0.0091	0.0448	0.1239	0.2633	0.4800	0.0908	0.1753	0.2328	0.2462	0.2550	0.9422	−0.5027
	0.50	0.0095	0.0466	0.1290	0.2743	0.5000	0.0936	0.1798	0.2374	0.2482	0.2411	0.9439	−0.4910
	0.53	0.0100	0.0494	0.1368	0.2907	0.5300	0.0977	0.1866	0.2440	0.2507	0.2210	0.9458	−0.4750
	0.55	0.0104	0.0513	0.1419	0.3017	0.5500	0.1005	0.1911	0.2482	0.2521	0.2080	0.9466	−0.4653
	0.58	0.0110	0.0541	0.1497	0.3182	0.5800	0.1047	0.1978	0.2544	0.2539	0.1892	0.9474	−0.4521
	0.60	0.0114	0.0560	0.1549	0.3291	0.6000	0.1075	0.2022	0.2585	0.2548	0.1771	0.9476	−0.4442

2.3. Left Censored Sample

Let $x_{(1)} < \ldots < x_{(n)}$ be an ordered sample of size n and $x_{(r_1)} < x_{(r_1+1)} < \ldots < x_{(n)}$ be the uncensored portion of size $(n-r_1+1)$ in which $x_{(r_1)}$ is the smallest available observation. The integer r_1 is the rank of the observation $x_{(r_1)}$ in the complete sample and is determined by a prefixed real number α such that $0<\alpha<1$, using the relation $r_1=[n\alpha]+1$. Then α is the proportion of censoring on the left.

For a given $k(<n-r_1+1)$, consider the sample quantiles $x_{(n_1)}, \ldots, x_{(n_k)}$ with ranks $n_1, \ldots, n_k$ determined by k fixed real numbers $\lambda_1, \ldots, \lambda_k$ such that $\alpha \leq \lambda_1 < \ldots < \lambda_k < 1$ and $n_i=[n\lambda_i]+1$, $i=1, \ldots, k$. As before define $\lambda_0=0$ and $\lambda_{k+1}=1$.

Based on k sample quantiles $x_{(u_1)}, \ldots, x_{(u_k)}$, the *ABLUE* of μ when σ is known and its variance are given by (1.4) and (1.5), respectively.

In order to determine the k optimum quantiles for the estimation of μ, we have to minimize $V(\mu^*)$ in (1.5) with respect to $u_1, \ldots, u_k$ over the domain $-\ln \ln(1/\alpha) \leq u_1 \leq \ldots \leq u_k \leq \infty$. Equivalently, we maximize K_1 in (2.2) on $\bar{S}(\alpha, 1)$. By Theorem 2.1, K_1 has a unique maximum at an interior point $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ of $\bar{S}(0, 1)$. For the maximum of K_1 on $\bar{S}(\alpha, 1)$ we have to consider the following two cases: (i) $\lambda_1^\circ \geq \alpha$ and (ii) $\lambda_1^\circ < \alpha$.

In the first case, we take the maximum of K_1 corresponding to $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ and the k optimum quantiles are the same as those in the complete sample.

For the second case, we prove the following theorem.

THEOREM 2.3: (i) The function K_1 in (2.2) defined on $\bar{S}(\alpha, 1)$ with $\lambda_1^\circ < \alpha \leq \lambda_1$ attains its maximum at a point $(\lambda_1'', \ldots, \lambda_k'')$ in $\bar{S}(\alpha, 1)$.

(ii) $(\lambda_1'', \ldots, \lambda_k'')$ is the unique solution of the system of equations

$$\frac{g_{i+1} - g_i}{\lambda_{i+1} - \lambda_i} + \frac{g_i - g_{i-1}}{\lambda_i - \lambda_{i-1}} - 2g_i' = 0 \qquad (i = 2, \ldots, k) \tag{2.11}$$

with $\alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_k < \lambda_{k+1} = 1$.

PROOF: (i) The proof is analogous to that of Theorem 2.2 (i).

(ii) From (i) we observe that the maximization of K_1 on $\bar{S}(\alpha, 1)$ is now equivalent to that of

$$\sum_{i=2}^{k+1} (g_i - g_{i-1})^2/(\lambda_i - \lambda_{i-1})$$

on $\bar{S}[\alpha, 1)$ with respect to $\lambda_2, \ldots, \lambda_k$. Following an argument analogous to the proof of Lemma 2.2, we see that $(\lambda_1'', \ldots, \lambda_k'')$ is a solution of (2.11).

To prove the uniqueness of solution of (2.11), let $t_i = \lambda_i/\lambda_{i+1}$ $(i = 1, \ldots, k)$ with $\lambda_1 = \alpha$. Clearly $0 < t_i < 1$ and $(g_{i+1} - g_i)/(\lambda_{i+1} - \lambda_i) = \ln \lambda_i - \ln t_i/(1 - t_i)$. Thus (2.11) reduces to $\ln t_i/(1 - t_i) + t_{i-1} \ln t_{i-1}/(1 - t_{i-1}) + 2 = 0$ which can be written as

$$h_1(t_i) = h_2(t_{i-1}) \qquad (i = 2, \ldots, k) \tag{2.12}$$

where $h_1(t)$ and $h_2(t)$ are as in Lemma 2.3.

Proceeding along similar lines as in the proof of Theorem 2.1, we can obtain a set of numbers $\{t_1'', \ldots, t_k''\}$ such that $t_1'' < t_2'' < \ldots < t_k''$. We can now determine λ_k'' uniquely by the equation $t_{k-1}'' = \lambda_k/\lambda_{k+1} = \lambda_k$, and then proceeding successively obtain the set $\{\lambda_1'', \ldots, \lambda_k''\}$ with $\lambda_k'' = \alpha$ such that $\lambda_1'' < \ldots < \lambda_k''$.

If the set $\{t_1'', \ldots, t_k''\}$ is not unique, let $\{t_1^*, \ldots, t_k^*\}$ be another set obtained from (2.12) with $t_1^* = \lambda_1^*/\lambda_2^* = \alpha/\lambda_2^*$, $t_2^* = \lambda_2^*/\lambda_3^*, \ldots, t_k^* = \lambda_k^*/\lambda_{k+1}^* = \lambda_k^*$. If we can show that $t_1'' = t_1^*$, then by Lemma 2.3, we must have $t_i'' = t_i^*$, $i = 2, \ldots, k$, and hence $\lambda_i'' = \lambda_i^*$, $i = 2, \ldots, k$. This in turn implies that (2.12) and hence (2.11) has a unique solution.

If $t_1'' \neq t_1^*$, let $t_1'' < t_1^*$. Therefore, since $\lambda_1'' = \lambda_1^* = \alpha$, we have $\lambda_2'' > \lambda_2^*$. But since $h_1(t) > h_2(t)$ for $0 \leq t < 1$, we have $t_i'' < t_i^*$, $i = 2, \ldots, k$ which implies $\lambda_i'' > \lambda_i^*$, $i = 2, \ldots, k$ and eventually $\lambda_{k+1}'' > \lambda_{k+1}^*$ which contradicts the relation $\lambda_{k+1}'' = \lambda_{k+1}^* = 1$. Hence we must have $t_1'' = t_1^*$. This completes the proof of the theorem.

REMARK 3: It may be observed from the proof of Theorem 2.3(ii) that if $(\alpha^{(3)}, \lambda_2^{(3)}, \ldots, \lambda_k^{(3)})$ and $(\alpha^{(4)}, \lambda_2^{(4)}, \ldots, \lambda_k^{(4)})$ are the optimum spacings for K_1 with left censoring at $\alpha^{(3)}$ and $\alpha^{(4)}$, respectively, and $\lambda_1^\circ \leq \alpha^{(3)} < \alpha^{(4)}$, then $\lambda_i^{(3)} < \lambda_i^{(4)}$ $(i=2, \ldots, k)$. For, let us consider the transformation

$$t_i^{(3)} = \lambda_i^{(3)}/\lambda_{i+1}^{(3)},\ t_i^{(4)} = \lambda_i^{(4)}/\lambda_{i+1}^{(4)} \qquad (i=2, \ldots, k) \tag{2.13}$$

with $\lambda_1^{(3)} = \alpha^{(3)}$, $\lambda_{k+1}^{(3)} = \lambda_{k+1}^{(4)} = 1$ as is done in the proof of Theorem 2.3(ii). Then

$$t_1^{(3)} \ldots t_k^{(3)} = \alpha^{(3)} < \alpha^{(4)} = t_1^{(4)} \ldots t_k^{(4)}.$$

It is obvious from Figure 1 and the preceding inequality that $t_i^{(3)} < t_i^{(4)}$ which implies $\lambda_i^{(3)} < \lambda_i^{(4)}$. In particular, if $\lambda_1^\circ = \alpha^{(3)}$ we have $\lambda_i^\circ < \lambda_i^{(4)}$ $(i=2, \ldots, k)$. Further, it is obvious from the relations $t_1^{(4)} \ldots t_k^{(4)} = \alpha^{(4)}$ and $t_k^{(4)} = \lambda_k^{(4)}$ that $\lambda_k^{(4)}$ increases to one with $\alpha^{(4)}$.

REMARK 4: Alternatively, if we know that $\lambda_k^{(3)} < \lambda_k^{(4)}$, $\alpha^{(3)} \geq \lambda_1^\circ$, and $\alpha^{(4)} \geq \lambda_1^\circ$, then (2.13) implies $\lambda_i^{(3)} < \lambda_i^{(4)}$ $(i=2, \ldots, k)$.

According to the definition (2.8) the ARE of the estimator μ^* is K_1.

The optimum spacing $\lambda_1'', \ldots, \lambda_k''$ with the corresponding coefficients $a_1, \ldots, a_k$, and the values of K_1 and K_3/K_1 for $k=2(1)5$ and various proportions of left censoring are presented in Table 3.

TABLE 3

Optimum spacings $\lambda_1'', \ldots, \lambda_k''$ with the corresponding coefficients and efficiencies for estimating μ based on $k=2(1)5$ sample quantiles from a left censored sample for various proportions of censoring α

k	α	λ_1	λ_2	λ_3	λ_4	λ_5	a_1	a_2	a_3	a_4	a_5	K_1	K_3/K_1
2	0.10	0.1000	0.3954				0.5193	0.4807				0.8160	−0.3971
	0.15	0.1500	0.4505				0.5915	0.4085				0.7932	−0.2862
	0.20	0.2000	0.4985				0.6457	0.3543				0.7603	−0.1789
	0.25	0.2500	0.5419				0.6896	0.3104				0.7218	−0.0732
	0.30	0.3000	0.5820				0.7266	0.2734				0.6798	0.0329
	0.35	0.3500	0.6195				0.7587	0.2413				0.6356	0.1408
	0.40	0.4000	0.6550				0.7872	0.2128				0.5898	0.2518
3	0.10	0.1000	0.2667	0.5556			0.4192	0.3365	0.2443			0.8621	−0.3136
	0.15	0.1500	0.3256	0.6015			0.4960	0.2995	0.2044			0.8245	−0.2138
	0.20	0.2000	0.3786	0.6400			0.5567	0.2683	0.1750			0.7822	−0.1159
	0.25	0.2500	0.4277	0.6740			0.6074	0.2410	0.1516			0.7374	−0.0181
	0.30	0.3000	0.4740	0.7047			0.6513	0.2165	0.1322			0.6910	0.0812
	0.35	0.3500	0.5182	0.7329			0.6902	0.1941	0.1156			0.6436	0.1830
	0.40	0.4000	0.5606	0.7593			0.7253	0.1736	0.1011			0.5955	0.2888
4	0.10	0.1000	0.2145	0.3932	0.6503		0.3747	0.2506	0.2278	0.1469		0.8785	−0.2811
	0.15	0.1500	0.2730	0.4493	0.6884		0.4526	0.2288	0.1966	0.1220		0.8356	−0.1856
	0.20	0.2000	0.3267	0.4978	0.7200		0.5156	0.2085	0.1722	0.1037		0.7900	−0.0913
	0.25	0.2500	0.3772	0.5415	0.7475		0.5690	0.1897	0.1519	0.0894		0.7429	0.0035
	0.30	0.3000	0.4254	0.5817	0.7721		0.6158	0.1722	0.1345	0.0775		0.6949	0.1000
	0.35	0.3500	0.4719	0.6193	0.7945		0.6576	0.1557	0.1191	0.0675		0.6464	0.1995
	0.40	0.4000	0.5170	0.6549	0.8153		0.6956	0.1403	0.1054	0.0588		0.5974	0.3032
5	0.10	0.1000	0.1867	0.3128	0.4855	0.7121	0.3494	0.1980	0.1935	0.1613	0.0979	0.8862	−0.2648
	0.15	0.1500	0.2444	0.3716	0.5367	0.7445	0.4277	0.1837	0.1707	0.1371	0.0808	0.8408	−0.1714
	0.20	0.2000	0.2980	0.4235	0.5801	0.7711	0.4918	0.1692	0.1518	0.1187	0.0685	0.7936	−0.0790
	0.25	0.2500	0.3489	0.4710	0.6185	0.7941	0.5467	0.1552	0.1356	0.1037	0.0588	0.7454	0.0142
	0.30	0.3000	0.3980	0.5152	0.6535	0.8145	0.5951	0.1417	0.1212	0.0911	0.0509	0.6968	0.1094
	0.35	0.3500	0.4456	0.5571	0.6859	0.8331	0.6386	0.1289	0.1082	0.0801	0.0442	0.6477	0.2077
	0.40	0.4000	0.4920	0.5971	0.7163	0.8503	0.6781	0.1166	0.0964	0.0705	0.0384	0.5984	0.3104

EXAMPLE: Let $k=3$, $n=95$, $\alpha=0.15$. We observe from Table 1 of [8] that this situation belongs to Case (ii). From Table 3 with $k=3$ and $\alpha=0.15$, it is easily seen that the optimum quantiles are $x_{(15)}$, $x_{(31)}$, and $x_{(58)}$. The ABLUE of μ is

$$\mu^* = 0.4960x_{(15)} + 0.2995x_{(31)} + 0.2044x_{(58)} + 0.2138\sigma$$

with $\mathrm{ARE}(\mu^*) = 82.45\%$.

2.4 Doubly Censored Sample

Let $x_{(1)} < \ldots x_{(n)}$ be an ordered sample of size n and $x_{(r_1)} < x_{(r_1+1)} < \ldots < x_{(r_2)}$ be the uncensored portion of size $(r_2 - r_1 + 1)$ in which $x_{(r_1)}$ and $x_{(r_2)}$ are, respectively, the smallest and largest available observations in the censored sample. The integers r_1 and r_2 are determined by two real numbers α and β such that $0 < \alpha < \beta < 1$ using the relations $r_1 = [n\alpha] + 1$ and $r_2 = [n\beta] + 1$. Then the quantities α and $1-\beta$ are, respectively, the proportions of censoring on the left and right, and we have a doubly censored sample.

For a given $k (< r_2 - r_1 + 1)$ consider the sample quantiles $x_{(n_1)} < \ldots < x_{(n_k)}$ with ranks $n_1, \ldots, n_k$ determined by k fixed real numbers $\lambda_1, \ldots, \lambda_k$ satisfying the order relation $\alpha \leq \lambda_1 < \ldots < \lambda_k \leq \beta$ and $n_i = [n\lambda_i] + 1$, $i = 1, \ldots, k$. Define $\lambda_0 = 0$ and $\lambda_{k+1} = 1$.

Based on k sample quantiles $x_{(n_1)}, \ldots, x_{(n_k)}$, the ABLUE of μ when σ is known and its variance are given by (1.4) and (1.5), respectively.

In order to determine the k optimum quantiles for the estimation of μ, we have to minimize $V(\mu^*)$ in (1.5) with respect to $u_1, \ldots, u_k$ over the domain $-\ln\ln(1/\alpha) \leq u_1 \leq \ldots \leq u_k \leq -\ln\ln(1/\beta)$. Equivalently, we maximize K_1 in (2.2) on $\bar{S}(\alpha, \beta)$. By Theorem 2.1, K_1 has a unique maximum at an interior point $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$ of $\bar{S}(0, 1)$. As before let $(\lambda_1', \ldots, \lambda_k')$ and $(\lambda_1'', \ldots, \lambda_k'')$ be the optimum spacings corresponding to right and left censoring, respectively. For the maximum of K_1 on $\bar{S}(\alpha, \beta)$, we have to consider the following five cases:

(i) $\alpha \leq \lambda_1^\circ, \beta \geq \lambda_k^\circ$,

(ii) $\alpha \leq \lambda_1', \beta = \lambda_k' < \lambda_k^\circ$,

(iii) $\alpha > \lambda_1', \beta = \lambda_k' < \lambda_k^\circ$,

(iv) $\lambda_1^\circ < \alpha = \lambda_1'', \beta \geq \lambda_k''$,

(v) $\lambda_1^\circ < \alpha = \lambda_1'', \beta < \lambda_k''$.

In cases (i), (ii), and (iv), we take the optimum spacings $(\lambda_1^\circ, \ldots, \lambda_k^\circ)$, $(\lambda_1', \ldots, \lambda_k')$, and $(\lambda_1'', \ldots, \lambda_k'')$, respectively, to which corresponds the maximum of K_1 defined on $\bar{S}(\alpha, \beta)$. Thus the optimum quantiles in Cases (i), (ii), and (iv) are the same as those in the complete, right censored and left censored samples, respectively.

In order to deal with Case (iii), we prove the following theorem.

THEOREM 2.4: Let $(\lambda_1', \ldots, \lambda_k')$ with $0 < \lambda_1' < \ldots < \lambda_k' = \beta$ be the optimum spacing for right censoring at $\beta < \lambda_k^\circ$, and suppose that $\alpha > \lambda_1'$. Then K_1 in (2.2) defined on $\bar{S}(\alpha, \beta)$ has a unique maximum at the point $(\bar{\lambda}_1, \ldots, \bar{\lambda}_k)$ in $S[\alpha, \beta]$ which is the unique solution of the system of equations

$$\frac{g_{i+1}-g_i}{\lambda_{i+1}-\lambda_i}+\frac{g_i-g_{i-1}}{\lambda_i-\lambda_{i-1}}-2g_i'=0 \quad (i=2, \ldots, k-1) \tag{2.14}$$

with $\alpha=\lambda_1<\lambda_2<\ldots<\lambda_{k-1}<\lambda_k=\beta$.

PROOF: Since K_1 defined on $\bar{S}(0,1)$ attains its maximum at an interior point of $\bar{S}(0,1)$, following similar arguments as in the proof of Theorem 2.2(i), K_1 attains its maximum at a boundary point $(\bar{\lambda}_1, \ldots, \bar{\lambda}_k)$ when defined on $\bar{S}(\alpha,\beta)$ if $\beta<\lambda_k^\circ$. As in the proof of Lemma 2.1, it can be shown that $\bar{\lambda}_1<\ldots<\bar{\lambda}_k$. So the maximum of K_1 occurs on the boundary of $S(\alpha,\beta)$ in the following three possible ways: (1) $\bar{\lambda}_1=\alpha$, $\bar{\lambda}_k=\beta$; (2) $\bar{\lambda}_1>\alpha$, $\bar{\lambda}_k=\beta$; and (3) $\bar{\lambda}_1=\alpha$, $\bar{\lambda}_k<\beta$.

For Case (1), we may write $K_1=K_1(\alpha,\lambda_2,\ldots,\lambda_{k-1},\beta)=Q_1+g^2(\alpha)/\alpha+g^2(\beta)/(1-\beta)$, where $Q_1=Q_1(\lambda_2,\ldots,\lambda_{k-1})=\sum_{i=2}^{k}(g_i-g_{i-1})^2/(\lambda_i-\lambda_{i-1})$ with $\lambda_1=\alpha$ and $\lambda_2=\beta$. Since $(\alpha,\bar{\lambda}_2,\ldots,\bar{\lambda}_{k-1},\beta)$ maximizes K_1, so $(\bar{\lambda}_2,\ldots,\bar{\lambda}_{k-1})$ maximizes Q_1. Further, since $(\bar{\lambda}_2,\ldots,\bar{\lambda}_{k-1})$ is an interior point of the domain $\alpha\leq\lambda_2\leq\ldots\leq\lambda_{k-1}\leq\beta$, following an argument similar to the proof of Lemma 2.2, we see that it must be the solution of (2.14). As in the proof of Theorem 2.1, we can solve (2.14) through the transformation $t_i=\lambda_i/\lambda_{i+1}$ $(i=1,\ldots,k-1)$ and obtain the set $\{\bar{t}_1,\ldots,\bar{t}_{k-1}\}$ such that $\bar{t}_1<\ldots<\bar{t}_{k-1}$. In order to show that this set is unique, we assume that there exists another set $\{t_1^*,\ldots,t_{k-1}^*\}$ with $\lambda_1^*=\alpha$ and $\lambda_k^*=\beta$ which corresponds to a solution of (2.14). Then we can obtain $t_i^*(i=1,\ldots,k-1)$ by the transformation $t_i^*=\lambda_i^*/\lambda_{i+1}^*$. If $\bar{t}_1\neq t_1^*$, let $\bar{t}_1<t_1^*$. Then it is obvious from Figure 1 that $\alpha/\beta=\bar{t}_1\ldots\bar{t}_{k-1}<t_1^*\ldots t_{k-1}^*=\alpha/\beta$ which is impossible. Thus the set $\{\bar{t}_1,\ldots\bar{t}_{k-1}\}$ is unique and hence $(\bar{\lambda}_1,\ldots,\bar{\lambda}_k)$ is the unique solution of (2.14)

For Case (2), we may write $K_1=K_1(\lambda_1,\ldots,\lambda_{k-1}\beta)=Q_2+g^2(\beta)/(1-\beta)$, where

$$Q_2=Q_2(\lambda_1,\ldots,\lambda_{k-1})=\sum_{i=1}^{k}(g_i-g_{i-1})^2/(\lambda_i-\lambda_{i-1})$$

with $\lambda_k=\beta$. Since for right censoring at β, the point $(\lambda_1',\ldots,\lambda_k')$ with $\lambda_k'=\beta$ maximizes $K_1(\lambda_1,\ldots,\lambda_k)$ on $\bar{S}(0,\beta)$, so $(\lambda_1',\ldots\lambda_{k-1}')$ maximizes Q_2 on the domain $0\leq\lambda_1\leq\ldots\leq\lambda_{k-1}\leq\beta$. Thus following an argument similar to the proof of Lemma 2.1, when defined on the above domain, Q_2 attains its maximum at an interior point $(\lambda_1',\ldots,\lambda_{k-1}')$ of the domain.

Since by assumption $\alpha>\lambda_1'$, following an argument similar to the proof of Theorem 2.3(i), Q_2 defined on the domain $\alpha\leq\lambda_1\leq\ldots\leq\lambda_{k-1}\leq\beta$ attains its maximum at a unique point, say, $(\hat{\lambda}_1,\ldots,\hat{\lambda}_{k-1})$ with $\alpha=\hat{\lambda}_1<\ldots<\hat{\lambda}_{k-1}<\beta$. So $(\alpha,\hat{\lambda}_2,\ldots,\hat{\lambda}_{k-1},\beta)$ a unique point maximizing $K_1(\lambda_1,\ldots,\lambda_{k-1},\beta)$ on $\bar{S}(\alpha,\beta)$. Thus $\hat{\lambda}_1=\alpha$ contradicts the assumption $\bar{\lambda}_1>\alpha$. Case (2) is therefore impossible.

For Case (3), we write $K_1=K_1(\alpha,\lambda_2,\ldots,\lambda_k)=g^2(\alpha)/\alpha+Q_3$, $\alpha\leq\lambda_2\leq\ldots\leq\lambda_k\leq\beta$, where $Q_3=Q_3(\lambda_2,\ldots,\lambda_k)=\sum_{i=2}^{k}(g_i-g_{i-1})^2/(\lambda_i-\lambda_{i-1})$ with $\lambda_1=\alpha$. Let us choose the point $\hat{\lambda}_k$ such that $(\alpha,\hat{\lambda}_2,\ldots,\hat{\lambda}_k)$ is the optimum spacing for $K_1(\lambda_1,\ldots,\lambda_k)$ on $\bar{S}(0,\beta)$ with right censoring at $\hat{\lambda}_k$. Since $\lambda_1'<\alpha$, by Remark 2, we have $\beta=\lambda_k'<\hat{\lambda}_k$. When considered as defined on $\alpha\leq\lambda_2\leq\ldots\leq\lambda_k\leq 1$, Q_3 also attains its maximum at the unique point $(\bar{\lambda}_2,\ldots,\bar{\lambda}_k)$. By similar arguments as in the proof of Lemma 2.1, this is an interior point of the above domain. Since $\beta<\hat{\lambda}_k$, when restricted to the domain $\alpha\leq\lambda_2\leq\ldots\leq\lambda_k\leq\beta$, Q_3 must attain its maximum at a point $(\lambda_2^*,\ldots,\lambda_k^*)$ with $\alpha<\lambda_2^*<\ldots<\lambda_k^*=\beta$, which in turn implies that $(\alpha,\lambda_2^*,\ldots,\lambda_{k-1}^*,\beta)$ maximizes K_1. So it cannot be that $\bar{\lambda}_k<\beta$. Case (3) is therefore impossible.

In order to deal with Case (v) we prove the following theorem.

THEOREM 2.5: Let $(\lambda_1'', \ldots, \lambda_k'')$ with $\alpha = \lambda_1'' < \ldots < \lambda_k'' < 1$ be the optimum spacing for left censoring at $\alpha > \lambda_1^\circ$, and suppose that $\beta < \lambda_k''$. Then K_1 in (2.2) defined on $\bar{S}(\alpha, \beta)$ has a unique maximum at the point $(\bar{\lambda}_1, \ldots, \bar{\lambda}_k)$ in $S[\alpha, \beta]$ which is the unique solution of the system of equations

$$\frac{g_{i+1} - g_i}{\lambda_{i+1} - \lambda_i} + \frac{g_i - g_{i-1}}{\lambda_i - \lambda_{i-1}} - 2g_i' = 0 \quad (i = 2, \ldots, k-1)$$

with $\alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_{k-1} < \lambda_k = \beta$.

PROOF: The proof is analogous to that of Theorem 2.4.

According to the definition (2.8) the *ARE* of the estimator μ^* is K_1.

TABLE 4

Optimum spacings $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$ with the corresponding coefficients and efficiencies for estimating μ based on $k = 2(1)5$ sample quantiles from doubly censored sample for various proportions of censoring at α and β

k	α	β	λ_1	λ_2	λ_3	λ_4	λ_5	a_1	a_2	a_3	a_4	a_5	K_1	K_3/K_1
2	0.10	0.30	0.1000	0.3000				0.4730	0.5270				0.8023	−0.4923
		0.35	0.1000	0.3500				0.4966	0.5034				0.8132	−0.4387
	0.20	0.35	0.2000	0.3500				0.5686	0.4317				0.7396	−0.2914
		0.45	0.2000	0.4500				0.6195	0.3805				0.7584	−0.2092
	0.30	0.45	0.3000	0.4500				0.6561	0.3439				0.6696	−0.0444
		0.55	0.3000	0.5500				0.7090	0.2910				0.6793	0.0181
	0.40	0.55	0.4000	0.5500				0.7309	0.2691				0.5856	0.2024
		0.65	0.4000	0.6500				0.7845	0.2155				0.5898	0.2501
3	0.10	0.35	0.1000	0.1998	0.3500			0.3818	0.2360	0.3823			0.8357	−0.4495
		0.55	0.1000	0.2649	0.5500			0.4180	0.3342	0.2478			0.8620	−0.3160
	0.20	0.40	0.2000	0.2886	0.4000			0.5074	0.1629	0.3297			0.7578	−0.2481
		0.60	0.2000	0.3643	0.6000			0.5475	0.2542	0.1983			0.7817	−0.1298
	0.30	0.45	0.3000	0.3700	0.4500			0.5965	0.1109	0.2926			0.6712	−0.0443
		0.70	0.3000	0.4722	0.7000			0.6502	0.2151	0.1347			0.6910	0.0800
	0.40	0.50	0.4000	0.4481	0.5000			0.6681	0.0691	0.2628			0.5803	0.1699
		0.75	0.4000	0.5568	0.7500			0.7229	0.1712	0.1059			0.5954	0.2868
4	0.10	0.35	0.1000	0.1614	0.2437	0.3500		0.3449	0.1543	0.1573	0.3435		0.8400	−0.4514
		0.65	0.1000	0.2144	0.3930	0.6500		0.3764	0.2506	0.2277	0.1470		0.8785	−0.2812
	0.20	0.40	0.2000	0.2566	0.3230	0.4000		0.4792	0.1099	0.1065	0.3044		0.7588	−0.2483
		0.70	0.2000	0.3227	0.4873	0.7000		0.5127	0.2035	0.1703	0.1135		0.7899	−0.0958
	0.30	0.45	0.3000	0.3455	0.3955	0.4500		0.5768	0.0755	0.0721	0.2756		0.6715	−0.0442
		0.75	0.3000	0.4205	0.5695	0.7500		0.6125	0.1670	0.1329	0.0876		0.6949	0.0957
	0.40	0.50	0.4000	0.4317	0.4650	0.5000		0.6559	0.0470	0.0450	0.2521		0.5804	0.1699
		0.80	0.4000	0.5133	0.6461	0.8000		0.6931	0.1369	0.1046	0.0654		0.5974	0.3007
5	0.10	0.45	0.1000	0.1573	0.2330	0.3297	0.4500	0.3299	0.1403	0.1437	0.1379	0.2482	0.8678	−0.3740
		0.70	0.1000	0.1855	0.3093	0.4784	0.7000	0.3483	0.1956	0.1916	0.1611	0.1034	0.8862	−0.2674
	0.20	0.50	0.2000	0.2585	0.3275	0.4077	0.5000	0.4684	0.1103	0.1065	0.0991	0.2157	0.7792	−0.1782
		0.75	0.2000	0.2951	0.4164	0.5670	0.7500	0.4897	0.1652	0.1492	0.1187	0.0772	0.7935	−0.0827
	0.30	0.55	0.3000	0.3533	0.4125	0.4779	0.5500	0.5704	0.0851	0.0803	0.0739	0.1902	0.6864	0.0208
		0.80	0.3000	0.3957	0.5099	0.6441	0.8000	0.5935	0.1391	0.1197	0.0913	0.0564	0.6967	0.1073
	0.40	0.60	0.4000	0.4450	0.4933	0.5449	0.6000	0.6535	0.0638	0.0597	0.0548	0.1682	0.5912	0.2316
		0.85	0.4000	0.4920	0.5970	0.7161	0.8500	0.6781	0.1165	0.0964	0.0705	0.0385	0.5984	0.3104

Table 4 provides the optimum spacing $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$ with the corresponding coefficients $a_1, \ldots, a_k$, and the values of K_1 and K_3/K_1 for $k=2(1)5$ and various proportions of double censoring.

In any given situation involving double censoring we proceed to use the tables as follows. Given two values of α and β, we first look at β. If for a given k:

(1) $\beta \geq \lambda_k^\circ$ in Table 1 of [8], then check whether $\alpha \leq \lambda_1^\circ$ or $\alpha > \lambda_1^\circ$. If $\alpha \leq \lambda_1^\circ$, the situation belongs to Case (i) and the optimum spacing is obtained from Table 1 of [8]. If $\alpha > \lambda_1^\circ$, then setting $\alpha = \lambda_1''$, we look at Table 3 and check whether $\beta \geq \lambda_k''$ or $\beta < \lambda_k''$. With the former, we have Case (iv) and the optimum spacing is obtained from Table 3. With the latter, we have Case (v) and the optimum spacing is obtained from Table 4.

(2) $\beta < \lambda_k^\circ$ in Table 1 of [8], then setting $\beta = \lambda_k'$, we look at Table 2 and check whether $\alpha \leq \lambda_1'$ or $\alpha > \lambda_1'$. With the former, we have Case (ii) and the optimum spacing is obtained from Table 2. With the latter, we have Case (iii) and Table 4 is used for the optimum spacing.

EXAMPLE 1: Let $n=106$, $k=3$, $\alpha=0.10$ and $\beta=0.35$. Since from Table 1 of [8] $\beta < \lambda_3^\circ = 0.4705$, setting $\lambda_3' = \beta$ for $k=3$ in Table 2, we have $\alpha > \lambda_1' = 0.0257$. This situation thus belongs to Case (iii). From Table 4 with the above choice of k, α, and β, the optimum quantiles are easily seen to be $x_{(11)}$, $x_{(22)}$, and $x_{(38)}$. The ABLUE of μ is

$$\mu^* = 0.3818x_{(11)} + 0.2360x_{(22)} + 0.3823x_{(38)} + 0.4495\sigma$$

with ARE $(\mu^*) = 83.57\%$.

EXAMPLE 2: Let $n=111$, $k=4$, $\alpha=0.40$ and $\beta=0.80$. Since from Table 1 of [8] $\beta > \lambda_4^\circ = 0.5486$ and $\alpha > \lambda_1^\circ = 0.0190$, setting $\lambda_1'' = \alpha$ for $k=4$ in Table 3, we have $\beta < \lambda_4'' = 0.8153$. This situation thus belongs to Case (v). The estimator μ^* can now be obtained from Table 4 as in Example 1.

2.5. Estimation of μ for the Special Case $k=1$

When $k=1$, K_1 in (2.2) reduces to

$$K_1 = K(\lambda_1) = \frac{g^2(\lambda_1)}{\lambda_1(1-\lambda_1)}.$$

By Theorem 2.1, K_1 has a unique solution λ_1° in the open interval (0, 1). This solution has been obtained numerically to be $\lambda_1^\circ = .2032$ correct up to four decimal places with the corresponding value of K_1 as 0.6476. Based on the spacing $\lambda_1 = 0.20319$, Lieblein [12] proposed an estimator with ARE 0.62.

Clearly, for right censoring at β with $\beta \geq \lambda_1^\circ$, the optimum spacing is λ_1°: otherwise it is β.

For left censoring at α with $\alpha \leq \lambda_1^\circ$, the optimum spacing is λ_1°; otherwise it is α.

For double censoring at α and β with $\alpha \leq \lambda_1^\circ \leq \beta$, the optimum spacing is λ_1°. The optimum spacings corresponding to $\alpha < \beta < \lambda_1^\circ$ and $\lambda_1^\circ < \alpha < \beta$ are β and α, respectively.

2.6 Comparison With the Maximum Likelihood Estimator

Harter and Moore [6] tabulated the asymptotic variances and covariances of the maximum likelihood estimators (MLE) of μ and/or σ of the Type I asymptotic distribution of smallest values based on all the available observations from censored samples. Their tables may be used for our extreme value distribution if we let their q_1 and q_2 be our $1-\beta$ and α, respectively. Table 5 compares the ARE of the MLE of μ when σ is known with the ARE given in Tables 2, 3, and 4 of our corresponding estimator μ^*. It may be observed from Table 5 that using no more than $k=5$ observations, μ^* has already

the ARE quite close to the ARE of MLE (especially for double censoring). The estimator μ^* thus seems to be a preferable one because of the simplicity in calculation of the coefficients, the high ARE even when the estimator is based on only a small number k of observations, and the uniqueness of the optimum spacings.

TABLE 5

Efficiencies of the MLE and the ABLUE of μ when σ is known

α	β	MLE	μ^*	α	β	MLE	μ^*
0	0.2	0.8476	0.7771 (2)[a]	0.1	0.3	0.8212	0.8023 (2)
0	0.3	0.9212	0.8673 (3)	0.2	0.4	0.7597	0.7578 (3)
0	0.4	0.9597	0.9161 (4)	0.3	0.7	0.6968	0.6910 (3)
0	0.5	0.9805	0.9439 (5)	0.4	0.5	0.5805	0.5803 (3)
0.2	0	0.8000	0.7822 (3)	0.4	0.8	0.5992	0.5974 (4)
0.3	0	0.7000	0.6949 (4)	0.3	0.8	0.6992	0.6967 (5)
0.4	0	0.6000	0.5984 (5)	0.4	0.6	0.5914	0.5912 (5)

[a] The figures in parentheses denote the value of k on which the estimator μ^* is based.

3. ESTIMATION OF σ WHEN μ IS KNOWN BASED ON k SAMPLE QUANTILES FROM A COMPLETE SAMPLE

The ABLUE of σ when μ is known based on k sample quantiles $x_{(u_1)}, \ldots, x_{(u_k)}$ and its variance are given by (1.6) and (1.7), respectively.

In order to determine the k optimum quantiles, we have to minimize $V(\sigma^*)$ in (1.7) with respect to $u_1, \ldots, u_k$ over the domain $-\infty \leq u_1 \leq \ldots \leq u_k \leq \infty$. Equivalently, we maximize K_2 which can be written in the alternative form

$$K_2 = K_2(u_1, \ldots, u_k) = \sum_{i=1}^{k+1} \frac{(u_i f_i - u_{i-1} f_{i-1})^2}{\lambda_i - \lambda_{i-1}}. \tag{3.1}$$

Using the transformation $u_i f_i = [-\ln \ln (1/\lambda_i)][-\lambda_i \ln \lambda_i] = q(\lambda_i) = q_i$, say, in (3.1), we have

$$K_2 = K_2(\lambda_1, \ldots, \lambda_k) = \sum_{i=1}^{k+1} \frac{(q_i - q_{i-1})^2}{\lambda_i - \lambda_{i-1}} \tag{3.2}$$

defined on $\bar{S}(0, 1)$.

We now prove the following theorem which will be needed in finding a spacing for the estimation of σ.

THEOREM 3.1: The system of equations

$$\frac{q_{i+1} - q_i}{\lambda_{i+1} - \lambda_i} + \frac{q_i - q_{i-1}}{\lambda_i - \lambda_{i-1}} - 2q_i' = 0 \quad (i = 1, \ldots, k), \tag{3.3}$$

where $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k < \lambda_{k+1} = 1$ and $q_i' = q'(\lambda_i) = dq(\lambda_i)/d\lambda_i$ has a solution which corresponds to the maximum of K_2 in (3.2) defined on $S(0, 1)$.

PROOF: The proof follows the same line of arguments as in the proofs of Lemmas 2.1 and 2.2 by replacing g_i by q_i.

The system of Eqs. (3.3) has been solved numerically by iterative procedure starting with a number of different sets of initial values. We have obtained thereby several solutions (by no means exhaustive) giving, in general, different values of K_2 which indicate that K_2 does not possess a unique maximum. Among the above solutions, we have chosen the one, say, $S=(\lambda_1^*, \ldots, \lambda_k^*)$ which yields the maximum value of K_2. For $k=6$, for example, twelve different sets of initial values were considered of which six produced the same solution giving a higher value of K_2 than any of the other six.

We observe from the proof of Theorem 3.1 that the value of K_2 in (3.2) increases with the number of λ_i's. So, as $\max |\lambda_i-\lambda_{i-1}|$ tends to zero,

$$\lim K_2(\lambda_1, \ldots, \lambda_k)=\lim \sum_{i=1}^{k+1} \left(\frac{q_i-q_{i-1}}{\lambda_i-\lambda_{i-1}}\right)^2 (\lambda_i-\lambda_{i-1})$$

$$=\int_0^1 \left(\frac{dq}{d\lambda}\right)^2 d\lambda=\int_0^1 \left(u\frac{f'}{f}\right)^2 dF-1=1.8235,$$

where f and F are given by (1.1) and (1.3), respectively, and $f'=df/du$. Thus the amount of information of the original sample with respect to σ is

$$I_0(\sigma)=1.8235\,\frac{n}{\sigma^2}.$$

So the Cramér-Rao lower bound for regular unbiased estimators of σ is $\sigma^2/(1.8235\,n)$.
We now define the ARE of the estimator σ^* as

$$\mathrm{ARE}(\sigma^*)=\frac{1/I_0(\sigma)}{V(\sigma^*)}=\frac{K_2}{1.8235}.$$

The spacing S chosen above has been compared with the following three other spacings with $i=1, \ldots, k$:

$$S_1 : \lambda_i=(i-\tfrac{1}{2})/k,$$

$$S_2 : \lambda_i=i/(k+1),$$

$$S_3 : \lambda_i=\lambda_i^\circ \text{ where } \lambda_i^\circ \text{ is as in Theorem 2.1.}$$

It has been observed that the spacing S gives higher efficiencies than any of the other three for at least $k=1, \ldots, 20$. Comparing S_1, S_2, and S_3 for $k=1, \ldots, 20$, we have found that S_3 yields highest efficiencies for $k=1, 7, \ldots, 20$ and S_1 for $k=2, \ldots, 6$. For comparison, we have presented in the last two rows of Table 6 ARE (σ^*) corresponding to S and ARE (σ^*) corresponding to the better of S_1 and S_3 for each $k=1(1)10$. The same table also gives the coefficients $b_1, \ldots, b_k$ and the values of K_2 and K_3/K_2 corresponding to the spacing S. Section 8 of [8] gives the values of the λ_i^*, b_i, K_3/K_1 and ARE (σ^*) for $k=2, 3, 4$. These are the same as the corresponding entries in Table 6 except perhaps a unit at the 4th decimal place.

EXAMPLE: Suppose $k=3$, $n=100$. From Table 6 with $k=3$, clearly the relevant quantiles for the estimation of σ are $x_{(2)}$, $x_{(7)}$, and $x_{(87)}$. The estimator of σ is therefore

$$\sigma^*=-0.1755x_{(2)}-0.3246x_{(7)}+0.2175x_{(87)}-0.2826\mu$$

with ARE $(\sigma^*)=74.82\%$.

TABLE 6

Spacings $\lambda_1^*, \ldots, \lambda_k^*$ with the corresponding coefficients and efficiencies for estimating σ based on $k=1(1)10$ sample quantiles from a complete sample

k	1	2	3	4	5	6	7	8	9	10
λ_1	0.0305	0.0373	0.0104	0.0111	0.0045	0.0047	0.0023	0.0024	0.0014	0.0014
b_1	−0.8000	−0.4521	−0.1755	−0.1671	−0.0836	−0.0829	−0.0478	−0.0478	−0.0302	−0.0303
λ_2		0.8678	0.0676	0.0730	0.0273	0.0283	0.0136	0.0139	0.0077	0.0078
b_2		0.2364	−0.3246	−0.3048	−0.1731	−0.1711	−0.1044	−0.1043	−0.0682	−0.0683
λ_3			0.8603	0.7799	0.0969	0.1011	0.0449	0.0459	0.0245	0.0249
b_3			0.2175	0.1776	−0.2337	−0.2291	−0.1566	−0.1561	−0.1073	−0.1074
λ_4				0.9497	0.7726	0.7220	0.1200	0.1233	0.0614	0.0624
b_4				0.0703	0.1742	0.1416	−0.1841	−0.1824	−0.1393	−0.1392
λ_5					0.9480	0.8910	0.7159	0.6809	0.1385	0.1412
b_5					0.0694	0.0837	0.1408	0.1172	−0.1515	−0.1506
λ_6						0.9758	0.8886	0.8411	0.6759	0.6501
b_6						0.0307	0.0836	0.0829	0.1170	0.0996
λ_7							0.9752	0.9381	0.8386	0.8002
b_7							0.0307	0.0455	0.0830	0.0781
λ_8								0.9866	0.9372	0.9002
b_8								0.0162	0.0456	0.0514
λ_9									0.9864	0.9616
b_9									0.0162	0.0273
λ_{10}										0.9918
b_{10}										0.0096
K_2	0.5986	1.1886	1.3643	1.5018	1.5755	1.6294	1.6671	1.6938	1.7156	1.7308
K_3/K_2	−0.8000	−0.2157	−0.2826	−0.2240	−0.2468	−0.2272	−0.2378	−0.2288	−0.2346	−0.2296
ARE(σ^*)	0.3283	0.6518	0.7482	0.8236	0.8640	0.8936	0.9143	0.9289	0.9408	0.9492
ARE(σ^*) : (S_1, S_3)	0.0771	0.3467	0.4976	0.5886	0.6500	0.6944	0.7293	0.7603	0.7851	0.8055

4. ESTIMATION OF μ AND σ BASED ON k SAMPLE QUANTILES FROM A COMPLETE SAMPLE

The ABLUE of μ and σ based on k sample quantiles $x_{(n_1)}, \ldots, x_{(n_k)}$ and the elements of their dispersion matrix with the generalized variance Λ are given by (1.8) and (1.9), respectively.

Let $I(\mu, \sigma)$ represent the information matrix for μ and σ defined by

$$I(\mu, \sigma) = n \begin{bmatrix} E\left(\frac{\partial \ln f}{\partial \mu}\right)^2 & E\left(\frac{\partial \ln f}{\partial \mu}\frac{\partial \ln f}{\partial \sigma}\right) \\ E\left(\frac{\partial \ln f}{\partial \mu}\frac{\partial \ln f}{\partial \sigma}\right) & E\left(\frac{\partial \ln f}{\partial \sigma}\right)^2 \end{bmatrix}$$

and let the inverse of $I(\mu, \sigma)$, denoted by Ω, be

$$\Omega = \frac{1}{n}\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix},$$

which corresponds to the Cramér-Rao lower bounds for regular unbiased joint estimators of μ and σ. Further, let D represent the determinant of Ω. Then the ARE of $\hat{\mu}$ and $\hat{\sigma}$, and their joint asymptotic efficiency (JAE) are defined by ([1], Section 32.6).

$$\text{(4.1)} \qquad \begin{cases} \text{ARE } (\hat{\mu}) = \dfrac{\Omega_{11}}{nV(\hat{\mu})} \\ \text{ARE } (\hat{\sigma}) = \dfrac{\Omega_{22}}{nV(\hat{\sigma})} \\ \text{JAE } (\hat{\mu}, \hat{\sigma}) = \dfrac{D}{\Lambda}. \end{cases}$$

By straightforward computation, we obtain

$$\text{(4.2)} \qquad \Omega = \frac{\sigma^2}{n}\begin{bmatrix} 1 + \dfrac{6(C-1)^2}{\pi^2} & \dfrac{6(C-1)}{\pi^2} \\ \dfrac{6(C-1)}{\pi^2} & \dfrac{6}{\pi^2} \end{bmatrix}, \quad D = \frac{6}{\pi^2}\frac{\sigma^4}{n^2}$$

where C denotes the Euler's constant defined by $C = 0.5772 \ldots$.

From (1.9) and (4.2), we obtain the efficiencies according to (4.1) as

$$\text{ARE } (\hat{\mu}) = \left[1 + \frac{6(C-1)^2}{\pi^2}\right]\frac{\Delta}{K_2}$$

$$\text{ARE } (\hat{\sigma}) = \frac{6}{\pi^2}\frac{\Delta}{K_1}$$

$$\text{JAE } (\hat{\mu}, \hat{\sigma}) = \frac{6}{\pi^2}.$$

We have considered the following four different spacings with $i=1, \ldots, k$:

T: $\lambda_i=\lambda_i^*$ where λ_i^* is as in the spacing S of section 3.

T_1: $\lambda_i=(i-\frac{1}{2})/k$,

T_2: $\lambda_i=i/(k+1)$,

T_3: $\lambda_i=\lambda_i^\circ$ where λ_i° is as in Theorem 2.1.

It has been observed that for at least $k=2, \ldots, 20$, the spacing T yields higher JAE $(\hat{\mu}, \hat{\sigma})$ and ARE $(\hat{\sigma})$ than any of the other three while the spacing T_1 gives higher ARE $(\hat{\mu})$. For $k=2(1)10$, the first three rows of Table 7 give all the efficiencies corresponding to the spacing T while the last row gives ARE $(\hat{\mu})$ for the spacing T_1.

TABLE 7

Efficiencies of the estimators $\hat{\mu}$ and $\hat{\sigma}$ based on the spacing of $\sigma^*(T)$ and the efficiencies of $\hat{\mu}$ based on the spacing $\lambda_i=(i-\frac{1}{2})/k(T_1)$ for $k=2(1)10$

k	2	3	4	5	6	7	8	9	10
JAE$(\hat{\mu}, \hat{\sigma})$:$T$	0.3340	0.4544	0.5862	0.6574	0.7220	0.7644	0.8006	0.8275	0.8499
ARE$(\hat{\sigma})$:T	0.6449	0.6914	0.8165	0.8397	0.8875	0.9002	0.9236	0.9313	0.9447
ARE$(\hat{\mu})$:T	0.5129	0.6079	0.7123	0.7615	0.8086	0.8368	0.8626	0.8803	0.8962
ARE$(\hat{\mu})$:T_1	0.7759	0.8656	0.9073	0.9307	0.9454	0.9554	0.9626	0.9679	0.9720

It is of interest to note that the ARE $(\hat{\mu})$ based on the spacing T_1 are quite close to the ARE $(\bar{\mu})$, where $\bar{\mu}$ is the MLE of μ (when σ is unknown) based on censored samples [6], if α and β are quite close to λ_1 and λ_k, respectively. For example, if $\alpha=0.1$ and $\beta=0.9$, ARE $(\bar{\mu})=0.9582$; for $k=5$, we have $\lambda_1=0.1$, $\lambda_5=0.9$, and ARE $(\hat{\mu})=0.9307$; for $k=6$, we have $\lambda_1=0.08$, $\lambda_6=0.92$, and ARE $(\hat{\mu})=0.9454$; if $\alpha=0.2$ and $\beta=0.8$, ARE $(\bar{\mu})=0.8819$; for $k=3$, we have $\lambda_1=0.17$, $\lambda_3=0.83$, and ARE $(\hat{\mu})=0.8656$.

Tiago de Oliveira [26] proposed unbiased moment estimators (Eq. (3.6′) of [26]) and simplified moment estimators (Eq. (3.1′) of [26]) of μ and σ which have the same joint asymptotic distribution. Downton [2] indicated that efficiencies of estimators linear in ordered observations are not particularly sensitive to changes in the coefficients. He proposed linear estimators of μ and σ with simple linear coefficients (Eqs. (3.4) and (3.5) of [2]) and with quadratic coefficients (Eqs. (4.20) and (4.21) of [2]). The ARE and JAE of the preceding estimators, which are based on all the n observations, are presented in Table 8 for comparison with Table 7. Looking at the simplified moment estimators, we find that using only $k=4$ observations with spacing T, our $\hat{\mu}$ and $\hat{\sigma}$ give higher JAE and ARE $(\hat{\sigma})$, while using $k=7$

TABLE 8

Efficiencies of Tiago de Oliveira's Simplified Moment Estimator and Downton's Linear Estimators with Linear and Quadratic Coefficients

	For μ	For σ	For (μ, σ)
Simplified moment estimator	0.9494	0.5526	0.4767
Linear estimator with linear coefficients	0.9963	0.7555	0.7211
Linear estimator with quadratic coefficients	0.9987	0.9364	0.9247

observations with spacing T_1, $\hat{\mu}$ yields higher ARE in estimating μ. Using $k=6$ observations with spacing T, JAE $(\hat{\mu}, \hat{\sigma})$ is about the same as the JAE of Downton's linear coefficient estimators, but we have higher ARE $(\hat{\sigma})$. The estimators with quadratic coefficients give quite high efficiencies, but such estimators use all the observations which is large when n is.

5. ESTIMATION OF THE SCALE AND SHAPE PARAMETERS OF THE WEIBULL DISTRIBUTION BASED ON k SAMPLE QUANTILES FROM A COMPLETE SAMPLE

Let y be a random variable having a two-parameter Weibull distribution with density function

$$\frac{c}{b}\left(\frac{y}{b}\right)^{c-1}\exp\left[-\left(\frac{y}{b}\right)^{c}\right], \qquad y \geq 0, \tag{5.1}$$

where b and c are, respectively, the unknown scale and shape parameters.

If we define the random variable $x=-\ln y$ and set $\mu=-\ln b$, $\sigma=1/c$, it is easily seen that x has the extreme value density function given by (1.1). Thus all the results of sections 2 to 4 apply for the estimation of μ and/or σ with the observations changed according to the above transformation. In order to indicate such application, we consider the following three cases.

CASE 1: Estimation of b when c is known.

Since μ^* is asymptotically normal with mean μ and variance $V(\mu^*)$ given in (1.5), by Theorem 6a.2(i) of [25] the estimator of b

$$b^*=\exp(-\mu^*)$$

is asymptotically normal with mean b and variance

$$V(b^*)=[\exp(-\mu)]^2V(\mu^*)=b^2V(\mu^*).$$

So Tables 1 and 3 of [8], 2, 3, and 4 can be used.

CASE 2: Estimation of c when b is known.

Since σ^* is asymptotically normal with mean σ and variance $V(\sigma^*)$ given in (1.6), the estimator of c

$$c^*=1/\sigma^*$$

is asymptotically normal with mean c and variance

$$V(c^*) = [-1/\sigma^2]^2 V(\sigma^*) = c^4 V(\sigma^*). \tag{5.2}$$

So Table 5 can be used.

CASE 3: Estimation of b and c when both are unknown.

Since the joint distribution of $\hat{\mu}$ and $\hat{\sigma}$ is asymptotically normal with means μ and σ and dispersion matrix given in (1.9), by Theorem 6a.2(iii) of [25] the joint distribution of the estimators of b and c

$$\hat{b} = \exp(-\hat{\mu}),\ \hat{c} = 1/\hat{\sigma}$$

is asymptotically normal with means b and c and dispersion matrix having elements

$$V(\hat{b}) = b^2 V(\hat{\mu}), V(\hat{c}) = c^4 V(\hat{\sigma}), \mathrm{Cov}(\hat{b}, \hat{c}) = bc^2 \mathrm{Cov}(\hat{\mu}, \hat{\sigma}) \tag{5.3}$$

So Table 6 can be used.

The MLE of b and/or c are asymptotically normal and efficient ([3, 10]), and the Cramér-Rao lower bounds for the regular unbiased estimators of b and/or c are the same as those for μ and/or σ except for some constant factors like b^2/c^2, c^2, b ([3], p. 127). So by (5.1), (5.2), and (5.3) the ARE and JAE of b^*, c^*, $\hat{b}$, $\hat{c}$ are the same as those for μ^*, σ^*, $\hat{\mu}$, $\hat{\sigma}$ given in Tables 1 and 3 of [8], 2 to 5, and 7.

Dubey [3] has proposed several large sample estimators of b and/or c based on selected sample quantiles. His single sample quantile estimator of b when c is known (Eq. (3.13) of [3]) is our b^* while that of c when b is known (Eq. (2.24) of [3]) with $\lambda_1 = 0.1121$ has an ARE of 0.2786 whereas our ARE $(c^*) = 0.3283$ when $k = 1$.

Menon [22] proposed a pair of estimators of b and c (Eqs. (3.3) and (5.1) of [22]) which can be obtained from Tiago de Oliveira's [26] simplified moment estimators by simple transformations. When b is known, Menon also proposed an estimator of c ([22], p. 179) using all the observations with ARE approximately 0.84 which is less than our ARE $(c^*) = 0.8640$ when $k = 5$.

6. ACCURACY OF THE TABLES

All computations were carried out by the IBM 7040 computer with all outputs to eight decimal places and they were rounded to four decimal places in the Tables of the present paper.

The values of the optimum spacing $\{\lambda_i^\circ\}$ for the ABLUE μ^* of μ based on complete samples were determined by the relation $t_i^\circ = \lambda_i^\circ/\lambda_{i+1}^\circ$ (cf. proof of Theorem 2.1) where the t_i° are presented in Table 1. When k is large, some accuracy may be lost due to the recursive iteration procedure. The values of $\{\lambda_i^\circ\}$, the coefficients $\{a_i\}$ in μ^*, K_1 and K_3/K_1 were computed for $k = 1(1)20$. They agree with those by Hassanein [8] up to five decimal places. So it is believed that the $\{\lambda_i^\circ\}$ computed by using Table 1 are quite accurate (at least up to five decimal places).

The optimum spacings in Tables 3, 4 and the spacings in Table 5 were obtained by solving numerically the systems of equations (2.11), (2.14) and (3.3), respectively. The method of iteration is used. An iteration procedure was stopped when the absolute value of errors between two consecutive solutions were less than 10^{-6}.

ACKNOWLEDGEMENT

The authors wish to express their gratitude to Mr. S. M. Shih for many useful suggestions and assistance in computations. They are also grateful to the referee for a number of helpful comments and suggestions on improvement in the exposition.

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ON ESTIMATION IN WEIBULL DISTRIBUTIONS WITH RANDOM SCALE PARAMETERS

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ABSTRACT

Work by the present authors on life distributions derived from stochastic hazard functions [4] is related to certain articles that have appeared in this journal. This relationship is illustrated. The emphasis of this article is upon problems of parameter estimation.

The purpose of this article is to draw attention to a recent paper, entitled "Life Distributions Derived from Stochastic Hazard Functions" [4], which was written by the authors of this paper and published elsewhere, and which in some ways is closely related to articles by Cozzolino [1] and Dubey [2] which have recently appeared in this journal, and to illustrate the applicability of the results that appeared in Ref. [4] to Refs. [1, 2]. In Ref. [2], "A Compound Weibull Distribution," Dubey has indicated the development of, and some results for a continuously-mixed Weibull distribution, while in Ref. [1] "Probabilistic Models of Decreasing Failure Rate Processes," Cozzolino has discussed the use of compound exponential distributions as models for decreasing failure rate processes. In the Harris-Singpurwalla paper, compound Weibull and compound exponential distributions are developed as special cases of lifetime distributions derived by assuming random hazard functions. In this more general approach, there are two problems of estimation: estimation of the parameters of the mixtures like those considered by Dubey and Cozzolino and estimation of the conditioning parameter of the conditional distributions. Moment estimators were considered in [4] and [2], but since maximum likelihood estimation (MLE) is not particularly more difficult for these problems, it will be used in this paper to illustrate the approach used in [4]. Also, similarly-derived estimators for the reliability will be given, in the event that the random variable can be interpreted as a lifetime.

In the discussions to follow, let the conditional distribution be referred to as the parent, and, in order to maintain continuity of development, the same notation as that of Dubey will be adopted here. Dubey has presented several forms of the parent Weibulls (Eqs. 6.1 through 6.5 in Ref. [2]) and the one which will be used in the following is his (6.4),

$$f(x|\beta) = (\alpha\beta)(x-\eta)^{\alpha-1} \exp\left[-\beta(x-\eta)^{\alpha}\right] \quad x > \eta \quad (\alpha, \beta > 0). \tag{1}$$

All subsequent results are obviously also valid for x exponential since the usual exponential density is a Weibull with $\alpha = 1$ and $\eta = 0$.

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In the event that the parent distribution has a parameter that is random, (say β), that is, that the hazard rate is considered to be a random function, then there are two types of estimation problems. For the first, suppose that *each* observation x_i is taken in accordance with a value β_i, where $\{\beta_i\}$ represents a sample from a population with density $f(\beta)$, and where $f(\beta)$ is known except for the value of the parameters γ and δ. Here one has first the probabilistic problem of deriving the unconditional distribution of x, $f(x)$, and then the statistical problem of estimating the parameters α, γ, δ, and η. Suppose instead that the observations $x_1, x_2, \ldots, x_n$ represent a random sample from a specific conditional distribution function $f(x|\beta)$; how does one estimate the β that determined the density from which the random sample was taken, given that β has the prior distribution $f(\beta)$, and that α, γ, δ, and η are known? This problem lends itself nicely to Bayesian estimation techniques since we have prior knowledge about the distribution of β.

To get an estimate of the first type here we derive the unconditional distribution and then proceed to obtain an estimate using some classical procedure. Dubey alludes to the derivation of moment-type and least-squares estimators of the parameters α, γ, δ, and η on pages 184 and 186 of [2], so here let us indicate a maximum likelihood procedure.

When the scale parameter β is treated as a random variable with the gamma density

$$f(\beta) = \delta^{\gamma}\beta^{\gamma-1}e^{-\delta\beta}/\Gamma(\gamma) \qquad \beta \geq 0\,(\delta, \gamma > 0) \tag{2}$$

and the parent is the three-parameter Weibull given by (1), the resulting compound Weibull distribution is easily shown to be

$$f(x) = \alpha\gamma\delta^{\gamma}(x-\eta)^{\alpha-1}/[\delta + (x-\eta)^{\alpha}]^{\gamma+1} \qquad x > \eta(\alpha, \gamma, \delta > 0). \tag{3}$$

(Eq. 5.4 of [2])

When such is the case, the compound or a posteriori failure rate is given by

$$\begin{aligned} h^*(x) &= f(x)/[1-F(x)] \\ &= \alpha\gamma(x-\eta)^{\alpha-1}/[\delta + (x-\eta)^{\alpha}]. \end{aligned}$$

From (3) the likelihood function is found to be

$$L(\alpha, \gamma, \delta, \eta; x) = (\alpha\gamma\delta^{\gamma})^{n} \prod_{i=1}^{n} \{[x_i-\eta]^{\alpha-1}/[\delta + (x_i-\eta)^{\alpha}]^{\gamma+1}\},$$

and the natural log of L to be

$$\begin{aligned} \ln L = n[\ln \alpha + \ln \gamma + \gamma \ln \delta] \\ + (\alpha-1)\sum_{i=1}^{n} \ln (x_i-\eta) \\ - (\gamma+1)\sum_{i=1}^{n} \ln [\delta + (x_i-\eta)^{\alpha}]. \end{aligned} \tag{4}$$

Note that the method of MLE will not work if the density $f(x)$ can go to ∞ for any allowable parameter values (as it *can* for $\alpha < 1$ and η not strictly less than all x_i). When such is not the case, the maximum likelihood estimates $\hat{\alpha}$, $\hat{\gamma}$, $\hat{\delta}$, and $\hat{\eta}$ would be given by the solution of the nonlinear equations

$$\begin{cases} 0 = n/\hat{\alpha} + \sum_{i=1}^{n} \ln (x_i - \hat{\eta}) - (\hat{\gamma}+1) \sum_{i=1}^{n} (x_i - \hat{\eta})^{\hat{\alpha}} \ln (x_i - \hat{\eta}) / [\delta + (x_i - \hat{\eta})^{\hat{\alpha}}] \\ 0 = n/\hat{\gamma} + n \ln \hat{\delta} - \sum_{i=1}^{n} \ln [\hat{\delta} + (x_i - \hat{\eta})^{\hat{\alpha}}] \\ 0 = n\hat{\gamma}/\hat{\delta} - (\hat{\gamma}+1) \sum_{i=1}^{n} \{1/[\hat{\delta} + (x_i - \hat{\eta})^{\hat{\alpha}}]\} \\ 0 = (\hat{\alpha}-1) \sum_{i=1}^{n} [-1/(x_i - \hat{\eta})] + (\hat{\gamma}+1) \sum_{i=1}^{n} \{(x_i - \hat{\eta})^{\hat{\alpha}-1} / [\hat{\delta} + (x_i - \hat{\eta})^{\hat{\alpha}}]\}. \end{cases}$$

The generalized Newton-Raphson method [5] may then be used for the solution of these four equations, bearing in mind the usual cautions about negative solutions, nonuniqueness of maxima, and finiteness.

For the Burr distribution, $\delta = 1$ and $\eta = 0$, and (4) becomes

$$\ln L = n[\ln \alpha + \ln \gamma] + (\alpha - 1) \sum_{i=1}^{n} \ln x_i - (\gamma + 1) \sum_{i=1}^{n} \ln [1 + x_i^{\alpha}].$$

The estimates are then the solution to

$$\begin{cases} 0 = n/\hat{\alpha} + \sum_{i=1}^{n} \ln x_i - (\hat{\gamma}+1) \sum_{i=1}^{n} [x_i^{\hat{\alpha}} \ln x_i / (1 + x_i^{\hat{\alpha}})] \\ 0 = n/\hat{\gamma} - \sum_{i=1}^{n} \ln (1 + x_i^{\hat{\alpha}}). \end{cases}$$

A first approximation, $p_1 = (\hat{\alpha}_1, \hat{\gamma}_1)$, could be obtained for example by the method of moments, and the next approximation would then be given by

$$\hat{\alpha}_2 = \hat{\alpha}_1 - [F(\hat{\alpha}_1)/F'(\hat{\alpha}_1)]$$

and

$$\hat{\gamma}_2 = n/[\sum_{i=1}^{n} \ln (1 + x_i^{\hat{\alpha}_2})],$$

where

$$F(\hat{\alpha}) = n/\hat{\alpha} + \sum_{i=1}^{n} \ln x_i - \left\{ n/\left[\sum_{i=1}^{n} \ln (1 + x_i^{\hat{\alpha}}) \right] + 1 \right\} \sum_{i=1}^{n} [x_i^{\hat{\alpha}} \ln x_i / (1 + x_i^{\hat{\alpha}})] - \hat{\alpha}.$$

The iterations would then be continued until two successive approximations are found to be sufficiently close to each other, and convergence to a root of $F(\hat{\alpha}) = 0$ is in fact guaranteed if (a) the initial approximate value of $\hat{\alpha}$, $\hat{\alpha}_1$, is sufficiently close to the root, (b) $F''(\hat{\alpha})$ does not become excessively large, and (c) $F'(\hat{\alpha})$ is never close to zero.

For the Lomax distribution (the same as that of Cozzolino in his Section 5), which is a form of Pareto [3] or inverted-beta type II distribution, $\alpha = 1$ and $\eta = 0$, and hence

$$\ln L = n[\ln \gamma + \gamma \ln \delta] - (\gamma + 1) \sum_{i=1}^{n} \ln (\delta + x_i).$$

Therefore the MLE's $\hat{\gamma}$ and $\hat{\delta}$ are given by the solution of the equations

$$\begin{cases} 0 = n/\hat{\gamma} + n \ln \hat{\delta} - \sum_{i=1}^{n} \ln (\hat{\delta} + x_i) \\ 0 = n\hat{\gamma}/\hat{\delta} - (\hat{\gamma} + 1) \sum_{i=1}^{n} [1/(\hat{\delta} + x_i)]. \end{cases}$$

If the first approximation is denoted by $p_1 = (\hat{\gamma}_1, \hat{\delta}_1)$, then the next one is given by

$$\hat{\delta}_2 = \hat{\delta}_1 - [F(\hat{\delta}_1)/F'(\hat{\delta}_1)]$$

and

$$\hat{\gamma}_2 = n \Big/ \left[\sum_{i=1}^{n} \ln (\hat{\delta}_2 + x_i) - \ln \hat{\delta}_2 \right],$$

where

$$F(\hat{\delta}) = n^2 \Big/ \left[\hat{\delta} \sum_{i=1}^{n} \ln (\hat{\delta} + x_i) - \hat{\delta} \ln \hat{\delta} \right] - \left\{ n \Big/ \left[\sum_{i=1}^{n} \ln (\hat{\delta} + x_i) - \ln \hat{\delta} \right] + 1 \right\} \sum_{i=1}^{n} [1/(\hat{\delta} + x_i)],$$

and so on.

For the second problem, suppose it is now assumed that α, γ, δ, and η are known and that $x_1, \ldots, x_n$ is a random sample from the Weibull density $f(x|\beta)$ given by (1), and it is desired to estimate the β that determined the density from which the random sample was taken. The parameter is chosen randomly according to a fully specified gamma prior, (2), and all observations are taken according to $f(x|\beta)$. This is a Bayesian estimation problem. If the loss incurred by choosing $\hat{\beta}$ when β is true is denoted by $d(\hat{\beta}, \beta)$ and the expected loss for fixed β by $R_{\beta}(\beta)$, then the Bayes estimator will be that function $\hat{\beta}(x_1, \ldots, x_n)$ which minimizes the overall expected loss $E[R_{\beta}(\beta)]$. When the loss is squared error, i.e., $d(\hat{\beta}, \beta) \equiv (\hat{\beta} - \beta)^2$, $\hat{\beta}$ is known to be the mean of the posterior distribution of β.

If the transformation $t = (x - \eta)^{\alpha}$ is made in (1), then the parent Weibull becomes

$$f(t|\beta) = \beta e^{-\beta t}.$$

The posterior distribution function of β is thus given by

$$G(\beta|t) = \left[\int_0^{\beta} (\beta)^{n+\gamma-1} e^{-\beta(n\bar{t}+\delta)} d\beta \right] \Big/ \left[\int_0^{\infty} (\beta)^{n+\gamma-1} e^{-\beta(n\bar{t}+\delta)} d\beta \right],$$

where t is the n-vector $(t_1, \ldots, t_n) = ((x_1 - \eta)^{\alpha}, \ldots, (x_n - \eta)^{\alpha})$ and $\bar{t}$ the mean. Therefore

$$\begin{aligned} \hat{\beta} = E_G[\beta] &= \left[\int_0^{\infty} (\beta)^{n+\gamma} e^{-\beta(n\bar{t}+\delta)} d\beta \right] \Big/ \left[\int_0^{\infty} (\beta)^{n+\gamma-1} e^{-\beta(n\bar{t}+\delta)} d\beta \right] \\ &= (n + \gamma + 1)/(n\bar{t} + \delta) \\ &= (n + \gamma + 1) \Big/ \left[\sum_{i=1}^{n} (x_i - \eta)^{\alpha} + \delta \right]. \end{aligned}$$

In the event that x is interpreted as a lifetime, the prior distribution $F(r(x_0))$ of the reliability $r(x_0)$ at a fixed time $x = x_0$ can be easily derived from the gamma prior on β by the appropriate trans-

formation on variables $R(x_0)=e^{-\beta(x_0-\eta)^\alpha}$, α and η are fixed. It can then be shown in a manner similar to that on p. 76 of [1] that the reliability estimate with respect to squared error is

$$\widehat{R(x)}=\left\{\left[\sum_{i=1}^{n}(x^t-\eta)^\alpha+\delta\right]\Big/\left[\sum_{i=1}^{n}(x_i-\eta)^\alpha+\delta+(x-\eta)^\alpha\right]\right\}^{n+\gamma}$$

For the Burr and Lomax cases the Bayes estimators become

$$\begin{cases}\hat{\beta}=(n+\gamma+1)\Big/\left[\sum_{i=1}^{n}x_i^\alpha+1\right]\\ \widehat{R(x)}=\left[\left(\sum_{i=1}^{n}x_i^\alpha+1\right)\Big/\left(\sum_{i=1}^{n}x_i^\alpha+1+x^\alpha\right)\right]^{n+\gamma}\end{cases}$$

and

$$\begin{cases}\hat{\beta}=(n+\gamma+1)\Big/\left[\sum_{i=1}^{n}x_i+\delta\right]\\ \widehat{R(x)}=\left[\left(\sum_{i=1}^{n}x_i+\delta\right)\Big/\left(\sum_{i=1}^{n}x_i+\delta+x\right)\right]^{n+\gamma},\end{cases}$$

respectively.

If (6.1) of [2] had been used instead of (6.4) as the form of the density of the parent Weibull and (5.3) as the density of β, then the two problems of estimation would be handled in similar fashion except that the results would be slightly different. For example, consider the Bayes problem. If the transformation $t=(x-\eta)^\alpha$ is made in the parent Weibull

$$f(x|\beta)=(\alpha/\beta)\ (x-\eta)^{\alpha-1}\exp[-(x-\eta)^\alpha/\beta],$$

then

$$f(t|\beta)=(1/\beta)e^{-t/\beta}.$$

The posterior distribution function of β is thus given by

$$G(\beta|t)=\left[\int_0^\beta(\beta)^{+(n+\gamma+1)}e^{-(n\bar{t}+\delta)/\beta}d\beta\right]\Big/\left[\int_0^\infty(\beta)^{-(n+\gamma+1)}e^{-(n\bar{t}+\delta)/\beta}d\beta\right].$$

Therefore

$$\begin{aligned}\hat{\beta}=E_{\hat{G}}[\beta]&=\left[\int_0^\infty(\beta)^{-(n+\gamma)}e^{-(n\bar{t}+\delta)/\beta}d\beta\right]\Big/\left[\int_0^\infty(\beta)^{-(n+\gamma+1)}e^{-(n\bar{t}+\delta)/\beta}d\beta\right]\\ &\overset{(\beta=1/\lambda)}{=}\left[\int_0^\infty\lambda^{n+\gamma-2}e^{-\lambda(n\bar{t}+\delta)}d\lambda\right]\Big/\left[\int_0^\infty\lambda^{n+\gamma-1}e^{-\lambda(n\bar{t}+\delta)}d\lambda\right]\\ &=\left[\sum_{i=1}^{n}(x_i-\eta)^\alpha+\delta\right]/(n+\gamma-1).\end{aligned}$$

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AN APPROXIMATIVE ALGORITHM FOR THE FIXED CHARGE PROBLEM

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ABSTRACT

This paper describes an approximate solution method for solving the fixed charge problem. This heuristic approach is applied to a set of test problems to explore the margin of error. The results indicate that the proposed fixed charge simplex algorithm is capable of finding optimal or near optimal solutions to moderate sized fixed charge problems. In the absence of an exact method, this heuristic should prove useful in solving this fundamental nonlinear programming problem.

INTRODUCTION

There exists an important class of problems in economics which can be expressed as a nonlinear programming problem of the form:

(1) $$\text{minimize } z=\sum_{j=1}^{n} (c_j x_j + k_j \delta_j)$$

subject to

(2) $$\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i=1, 2, \ldots, m$$

$$x_j \geq 0 \text{ for } j=1, 2, \ldots, n,$$

where

$$\delta_j = \begin{cases} 0 \text{ if } x_j = 0 \\ 1 \text{ if } x_j > 0 \end{cases}.$$

Problems with this nonlinear objective function and linear constraints have traditionally been called the *fixed charge problem* [5].

In practice fixed charges are frequently incurred whenever the initiation of an economic activity at a positive level causes the firm to incur a lump sum charge. The magnitude of the fixed charges are assumed to be in no way influenced by the level of the initiated activity. The cost function for each economic activity can then be represented by Figure 1.

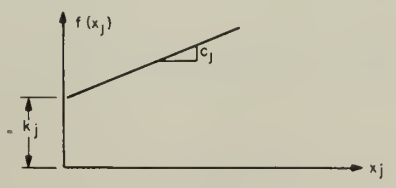

FIGURE 1

Fixed charge problems have proven difficult to solve by exact mathematical programming methods. Approximative techniques capable of solving the fixed-charge transportation problem have been developed by Kuhn and Baumol [6] and by Balinski [1]. In theory, the fixed charge problem can be formulated as an integer programming problem. Unfortunately, existing integer programming codes do not appear to be capable of effectively solving fixed charge problems. Recently, two approximate solution methods for solving the fixed charge problem were developed by Cooper and Drebes [2]. It is the purpose of this paper to report on another approximative algorithm for solving the fixed charge problem.

CHARACTERISTICS OF FIXED CHARGE PROBLEMS

It is possible to show that the global optimum solution to the fixed charge problem will occur at an extreme point of the convex set of feasible solutions specified by (2). This can be done by first noting that the objective function of the fixed charge problem is a concave function over a convex set of feasible solution.* It is also known that the global minimum of a concave function will be taken on at an extreme point of a closed convex set which is bounded from below (see [4], page 93). Thus in seeking to determine the global optimal solution of the fixed charge problem, it is only necessary to examine the extreme point solutions of the convex set of feasible solutions.

Under certain conditions, it is possible to solve the fixed charge problem as a linear programming problem. Clearly, if all fixed charges were equal to zero, the problem's objective function would be linear. Hirsch and Dantzig [5] proved that if the fixed charge associated with each economic activity had the same nonnegative magnitude, then in the absence of degeneracy, the optimal solution to the corresponding linear programming problem, obtained by ignoring all fixed charges, is also the optimal solution to the fixed charge problem. It is necessary to assume the absence of degeneracy since a degenerate basic feasible solution will incur less than m fixed charges and thus might be a better solution than the optimal solution obtained by solving the linear programming problem.

At present, there exists no mathematically exact method capable of determining the global optima of fixed charge problems. For small problems, it is possible to enumerate all basic solutions and select that basic feasible solution with the lowest value. For most real problems, however, this approach is not computationally feasible since the number of basic solutions increases approximately at a combinatorial rate.*

Fixed charge problems have proven difficult to solve by exact mathematical programming methods because their nature is such that many local minima exist. If each activity variable, x_j, has a positive fixed charge associated with it, each extreme point will be a local minima. Any nonbasic feasible solution in an ϵ-neighborhood about an extreme point would be inferior to that extreme point's solution value.

The problem of many local minima may be overcome by using an adjacent extreme point method like that used in the simplex method. It is possible, however, that a second type of local minima exists which is superior to all adjacent extreme point solutions, but inferior to the global minimum. That is, there are basic feasible solutions that cannot be improved by replacing only one vector in the solution with a vector not currently in the basis. Thus when adjacent extreme point methods are applied to fixed charge problems, the algorithm need not terminate at the global optimal solution.

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*For a concise proof of this statement, see [2], page 104.

*The maximum number of basic solutions to a problem with n variables and m constraints is $\binom{n}{m}$.

From this point on, the term local minima will refer to this second type of local minima since the heuristic method reported here uses an adjacent extreme point approach.

THE FIXED CHARGE SIMPLEX ALGORITHM

The following approximative algorithm is proposed as a practical method for solving the fixed charge problem as defined by (1) and (2). It is called the fixed charge simplex algorithm because it retains most of the decision rules of Dantzig's simplex method while also taking into account the fixed charges associated with the vector entering and the vector leaving the basis at each iteration. It differs from simplex theory only in that a modified criterion is used to select the vector to enter the basis. Since the nonnegativity constraints on the x_j variables will be observed, the vector selected to leave the basis will still be determined in the same manner as in the simplex method.

In the proposed fixed charge simplex algorithm, the criterion used to select the vector to enter the basis takes into account both the variable costs and the fixed charges. It selects that vector not in the basis which offers the greatest improvement in the objective function if brought into the basis in the next iteration. Once the entering vector has been selected, the fixed charge simplex algorithm reverts to standard linear programming procedures for transforming each tableau.

When optimality has been reached, there should exist no nonbasic vector that could replace a basic vector and result in an improved value of the objective function; however, the optimal solution obtained may be a local minima since the simplex method only permits adjacent extreme point solutions to be evaluated at each iteration. Thus the fixed charge simplex algorithm can only be considered a heuristic approach.

The basic elements of the proposed fixed charge simplex algorithm are:

STEP 1. Determine if there exists a basic feasible solution to the linear programming problem that is obtained if all fixed charges are ignored. If no basic feasible solution to the LP problem exists, then the program terminates since no feasible solution will exist for the fixed charge problem.

STEP 2. Starting with the initial basic feasible solution obtained in STEP 1, determine for each nonbasic vector its $z_j - c_j$ in the same way as in linear programming.

STEP 3. For each nonbasic vector, determine the vector which would have to leave in order to maintain feasibility if that vector were to be inserted into the solution. The standard simplex criterion is used here:

$$\theta_j = \min\left\{\frac{x_{Bi}}{y_{ij}}, \text{ for } y_{ij} > 0\right\}, \tag{3}$$

where x_{Bi} is the i-th component of the basic solution vector, and y_{ij} is the component common to row i and column j of the standard simplex tableau.

STEP 4. For each nonbasic vector, determine the total change in variable costs (excluding fixed charges) would result if that vector were introduced into the basis,

$$d_j = \theta_j(z_j - c_j) \tag{4}$$

where d_j is the total change in variable costs.

STEP 5. For each nonbasic vector, determine the change in total fixed charges that would result if that vector were introduced into the basis.

$$e_j = k_{r_j} - k_j, \tag{5}$$

where k_j is the fixed charge associated with vector j, and k_{r_j} is the fixed charge associated with the basic vector which would have to leave the basis if vector j were to enter.

STEP 6. For each nonbasic vector, determine the total change in the objective function that would result if that vector were to be introduced into the basis

$$s_j = d_j + e_j, \tag{6}$$

where s_j is the total change in the objective function.

STEP 7. If all $s_j \leq 0$, go to STEP 9 since a local minimum has been located. If one or more $s_j > 0$, select the most positive s_j to enter the basis during the next iteration.

STEP 8. Transform the matrix using standard simplex procedures. Then go to STEP 3.

STEP 9. Record and print out optimal solution.

STEP 10. Randomly select a nonbasic variable to enter the basis.

STEP 11. Determine basic variable which must leave if feasibility is to be maintained.

STEP 12. Transform matrix and compute the s_j for each nonbasic variable.

STEP 13. If all adjacent extreme point solutions are inferior to the best known local optimal solution, go to the next step. If an adjacent extreme point solution is superior to the best solution, proceed to that extreme point and go to STEP 3.

STEP 14. If the random walk phase has gone L iterations without improving the solution, the search phase is terminated. Otherwise, go to STEP 10.

The initial phase of the fixed charge simplex algorithm contained only the first nine steps. The last five steps were added after the initial phase of testing.

THE TEST PROBLEM SET

In order to test the effectiveness of the proposed fixed charge simplex algorithm, a test problem set was generated. Fixed charge problems were randomly generated by using the same problem generator subroutine developed in the study by Cooper and Drebes [2]. The global minima for each problem in their test problem set had previously been determined by evaluating all basic feasible solutions for each problem. Unfortunately, the computational limitations of this combinatorial approach restricted the size of the test problems to (5×10) and (7×15) problems.

Using the Cooper and Drebes' problem generator, 200 (5×10) problems were constructed with a matrix density of 40 percent. Sixty-six (7×15) problems were also constructed with a matrix density of 30 percent. A check indicated that these problems were identical to those used in the Cooper and Drebes' study.

This study also constructed large problems by combining a series of smaller problems as follows:

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

where A_1, A_2, and A_3 are the matrices of smaller problems whose global minima are known. Since there can be no interaction between the individual smaller problems, the global optimal solution of the larger problem is the sum of the optimal solutions of the three smaller problems. By utilizing this approach, 22 (15×30) test problems were constructed from 66 previously generated (5×10) test problems.

THE TEST RESULTS

The initial version fixed charge simplex algorithm was applied to the randomly generated test problem set. The overall results of the first phase of testing is given in the M–1 rows of Table 1. While the basic version of the algorithm, (M–1), was not as effective in finding the global optimal solutions as the methods developed by Cooper and Drebes, the computational efficiency of this version provided a sufficient operation time advantage to justify the addition of certain postoptimality tests designed to search for the existence of a superior solution.

It was decided to add an adjacent extreme point random walk phase which began when the fixed charge simplex algorithm had reached a local optimal extreme point. A nonbasic vector is randomly selected to enter the basis. The vector selected to leave the basis is chosen according to the standard simplex criterion, in order to maintain feasibility. At each new extreme point, the algorithm examines all adjacent extreme point solutions. If an improved solution is found, the algorithm proceeds to that extreme point and reverts back to the fixed charge simplex method phase. If no adjacent extreme points have lower solutions, then another nonbasic vector is randomly introduced until either an improved solution is found or the upper iteration limit of the random walk phase has been reached. For test purposes, the upper iteration limit was set at $2n$ in the M–2 version and at $3n$ in the M–3 version, where n is the number of variables in the test problem.

TABLE 1

Algorithm Test Results

Problem size	Model	Number tested	Number optimal	Percent optimal (%)	Average deviation from global optimality (%)	Average IBM 7072 time: time per problem (sec.)
(5 × 10)	M–1	200	169	85	1.27	1.6
	M–2	200	196	98	0.13	7.0
	M–3	200	200	100	0.	14.0
(7 × 15)	M–1	66	51	77	2.0	3.0
	M–2	66	59	89	0.74	12.0
	M–3	66	66	100	0.	26.0
(15 × 30)	M–1	22	14	64	1.0	.....
	M–3	22	17	77	0.58	75

The M–2 and M–3 versions of the fixed charge simplex algorithm were applied to the same set of test problems. The overall results are given in the M–2 and M–3 rows of Table 1. The improvement in the algorithm's performance was achieved at the expense of its computational efficiency. These times, however, still are considerably less than the times required by the heuristic methods developed by Cooper and Drebes (see [2], page 104).

In M–1 version of the fixed charge simplex algorithm was also tested on the following example problem which was given on page 141 of [4]:

$$\min z = 2\sum_{j=1}^{7} y_j + 300\sum_{j=1}^{7} \delta_j \tag{7}$$

subject to

$$
\begin{aligned}
& x_1 - y_1 = 90 \\
& y_j + x_j - y_{j+1} = d_j \qquad j = 2, \ldots, 6 \\
& x_7 + y_7 = 130 \\
& \\
& x_j \geq 0 \qquad y_j \geq 0 \qquad j = 1, 2, \ldots, 7,
\end{aligned} \tag{8}
$$

where $d_2 = 125$, $d_3 = 140$, $d_4 = 100$, $d_5 = 45$, and $d_6 = 60$. The M–1 version found a solution of 1780 which is the second best solution. The M–2 and M–3 versions found a solution of 1770 which is identical to the global optimal solution obtained by a dynamic programming approach.

IN CONCLUSION

The results of this study suggest that the initial version of the fixed charge simplex algorithm is able to locate the global optimal solution of most fixed charge problems. Its major shortcoming is that the algorithm occasionally terminates at extreme points with local minima which are considerably inferior to the global minima. The addition of the random search phase to the algorithm appears capable of reducing the likelihood that the algorithm will terminate at an inferior local minimum. The effectiveness of this phase will quite likely decrease as the size of the problem's matrix is increased. In the absence of an exact mathematical programming method, however, this heuristic should prove useful when applied to economic problems.

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A QUADRATIC ASSIGNMENT PROBLEM WITHOUT COLUMN CONSTRAINTS

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ABSTRACT

We convert a quadratic assignment problem [1] with a nonconvex objective function into an integer linear program. We then solve the equivalent integer program by a simple enumeration that produces global minima.

We consider the problem (a variation of the quadratic assignment problem):

$$\text{minimize} \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} a_{ik} x_{ij} x_{kj}$$

(1)

$$\text{when} \sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, \ldots, m)$$

$$x_{ij} = 0 \text{ or } 1, \quad (i = 1, \ldots, m; j = 1, \ldots, n)$$

where $m > n$ and $A = (a_{ik})$ is a given symmetric matrix with nonnegative elements and zero main diagonal terms. Note that (1) is the standard quadratic assignment problem with the column constraints omitted.

For a discussion of problem (1) see [1], where an algorithm is presented that gives a local minimum. Global minima are not always produced by the method of [1] because the objective function in (1) is not convex. In this paper, we convert (1) to an integer program. We then present an algorithm that produces the global minima by a simple enumeration.

THE EQUIVALENT INTEGER PROGRAM

We make the change of variables

$$y_{ik} = \sum_{j=1}^{n} x_{ij} x_{kj}, \qquad (i = 1, \ldots, m-1; k = i+1, \ldots, m). \tag{2}$$

We are interested in converting (1) to a problem containing the y_{ik} alone. Once the y_{ik} are found from the equivalent problem, we must show how to determine the x_{ij}.

Consider the m by n matrix $X = (x_{ij})$ having elements zero or one. We make the following observations:

1. The linear constraints in (1) allow only a single one to appear in each row of X.

2. We obtain $y_{st} = 1$ only when $x_{sj} = 1$ and $x_{tj} = 1$ for $s < t$ and a single j value. Thus the objective function in (1) can have a nonzero value only when X has two or more unit elements in the same column.

3. If d elements of a column of X have unity value then at least $\binom{d}{2}$ elements y_{ik} have unity value. (By $\binom{d}{2}$ we mean two combinations out of d and $\binom{d}{2}=\frac{d(d-1)}{2}$.)

4. If $y_{st}=1$ and $y_{rt}=1$ for $s<r<t$ then $y_{sr}=1$. This is true because $y_{st}=1$ only if $x_{sj}=1$ and $x_{tj}=1$, and $y_{rt}=1$ only if $x_{rj}=1$ and $x_{tj}=1$. Thus, since $x_{sj}=1$ and $x_{rj}=1$ we must have $y_{sr}=1$. Similarly, if $y_{st}=1$ and $y_{tr}=1$ for $s<t<r$, then $y_{sr}=1$.

5. The columns of X may be interchanged in any way and will still produce the same values for y_{ik} and the objective function in (1).

From these observations, we can think of problem (1) as one of loading zeroes and ones into matrix X and calculating the effect on the y_{ik} and in turn the objective function in (1). The major constraint on the y_{ik} is determined by the assignments in X. This constraint is obtained by finding the loading of X that minimizes $\sum_{i=1}^{m-1}\sum_{k=i+1}^{m} y_{ik}$, which is obtained by attempting to equalize the number of ones in each column. We write $m=nd+m_0$, where $d=\left[\frac{m}{n}\right]$ and $m_0=(\text{mod } n)$; $[x]$ means the greatest integer less than or equal to x and mod n is the usual remaindering operation.

A solution that minimizes $\sum_{i=1}^{m-1}\sum_{k=i+1}^{m} y_{ik}$ will have $(n-m_0)$ columns with d ones and m_0 columns with $d+1$ ones. Thus

$$\min \sum_{i=1}^{m-1}\sum_{k=i+1}^{m} y_{ik}=n\binom{d}{2}+dm_0.$$

In general, we have the constraint

$$\sum_{i=1}^{m-1}\sum_{k=i+1}^{m} y_{ik}\geq n\binom{d}{2}+dm_0. \tag{3}$$

The integer program equivalent to (1) is

$$\text{minimize } \sum_{i=1}^{m-1}\sum_{k=i+1}^{m} a_{ik}y_{ik}, \tag{4}$$

subject to $y_{ik}=0$ or 1, the results of the fourth observation, and the constraint (3). Once the y_{ik} are found the x_{ij} are obtained by loading the matrix X depending on which y_{ik} values are unity. The observations above tell us the values of x_{ij} to set to one; in particular, the fifth observation indicates that there is considerable freedom in doing so. The major constraint is (3) and although the quantification of the fourth observation requires many constraints, they may be included in a simple enumeration to solve the equivalent integer program. Note that the integer program uses the terms above the main diagonal of A and has a solution value that is one-half that for (1). The integer program is solved in the following algorithm:

We define ϕ as the null set and $|T|$ as the number of elements in set T. We also have $d=\left[\frac{m}{n}\right]$, $m_0=m \pmod n$, and $r=n\left(\frac{d}{2}\right)+dm_0$.

1. List the values of the problem as

	1	2	3	. . .	m
1		a_{12}	a_{13}	...	a_{1m}
2			a_{23}	...	a_{2m}
.				...	.
.				...	.
.				...	.
$m-1$					$a_{m-1,\,m}$;

set $\bar{s}=0$ for this list. Go to 2.

2. Given lists of the form

$\bar{s}$	1	2	3	. . .	m
$\bar{T}_1$	$\bar{a}_{11}$	$\bar{a}_{12}$	$\bar{a}_{13}$	. . .	$\bar{a}_{1m}$
$\bar{T}_2$	$\bar{a}_{21}$	$\bar{a}_{22}$	$\bar{a}_{23}$	. . .	$\bar{a}_{2m}$
.	.	.	.	. . .	.
.	.	.	.	. . .	.
.	.	.	.	. . .	.
$\bar{T}_{m-1}$	$\bar{a}_{m-1,\,1}$	$\bar{a}_{m-1,\,2}$	$\bar{a}_{m-1,\,3}$	. . .	$\bar{a}_{m-1,\,m}$

find $\bar{a}_{ef}=\min \bar{a}_{ij}$ for all unmarked elements on all lists. Mark the element. Set $s=\bar{s}$, $T_e=\bar{T}_e$ and go to 3.

3. If $s+|T_e|\geq r$ go to 4. Otherwise form an additional list with values obtained from the newly marked list as follows:

(a) Replace the $\bar{T}_e$ heading with $\bar{T}_e=T_e\cup f$.

(b) Delete the f column and the T_e column if it exists. Delete the f row if it exists.

(c) Set $\bar{s}=s+|T_e|$.

(d) Elements in the $\bar{T}_e$ row are $\bar{a}_{eh}=\bar{a}_{ef}+\sum_i a_{ih}$ for $i\epsilon\bar{T}_e$ and where $a_{ih}=a_{hi}$ for $i>h$. The indices h are the column headings. The remaining elements are $\bar{a}_{ij}=\bar{a}_{ef}+\bar{a}_{ij}$. Go to 2.

4. The problem is solved with objective function value $\bar{a}_{ef}$. Using the newly marked list, we find the x_{ij} by loading the X matrix as follows:

(a) Replace $\bar{T}_e$ by $\bar{T}_e=T_e\cup f$.

(b) Set $k=1$, $j=1$ and go to 4(c).

(c) Set $x_{ij}=1$ for $i\epsilon\bar{T}_k$; set $k=k+1$ and go to 4(d).

(d) If $\bar{T}_k=\emptyset$, go to 4(e). Otherwise set $j=j+1$ and go to 4(c).

(e) If an assignment has been made in row m, end; otherwise set $x_{mu}=1$ and end.

This completes the algorithm. The method implicitly contains an enumeration of all possible values of the y_{ik} that satisfy the fourth observation in order of increasing values of the objective function (4). We stop when constraint (3) is satisfied. The corresponding x_{ij} are then produced and represent the solution to the global minimum of problem (1). This is seen as follows: suppose the values of the y_{ik} and x_{ij} produced by the algorithm are y_{ik}^0 and x_{ij}^0 and there exist $\bar{x}_{ij}$ that satisfy the constraints in (1) with

$$\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} a_{ik}\bar{x}_{ij}\bar{x}_{kj} < \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} a_{ik}x_{ij}^0x_{kj}^0.$$

If we calculate

$$\bar{y}_{ik} = \sum_{j=1}^{n} \bar{x}_{ij}\bar{x}_{kj}$$

as in (2), we have

$$\sum_{i=1}^{m-1}\sum_{k=i+1}^{m} a_{ik}\bar{y}_{ik} < \sum_{i=1}^{m-1}\sum_{k=i+1}^{m} a_{ik}y_{ik}^{0},$$

where the $\bar{y}_{ik}$ satisfy the fourth observation. If the $\bar{y}_{ik}$ also satisfy (3), we would have a contradiction since the y_{ik}^{0} produce the minimum value in (4); however, the right side of (3) is the smallest possible value for

$$\sum_{i=1}^{m-1}\sum_{k=i+1}^{m} y_{ik}$$

consonant with (2). Thus (3) is satisfied by the $\bar{y}_{ik}$ and the x_{ij}^{0} must be the global minimum in (1).

Care must be taken in computational work using the algorithm since storage does become a problem; however, the lists developed reduce in size as the algorithm progresses. Lists, beyond the first, with all marked elements may be discarded. Further, redundant information may develop in several lists and may be discarded; however, the number of lists grows rapidly and storage may be a problem even for moderate-sized problems.

We use as example the problem in [1]. Step 1 of the algorithm defines the problem:

(5)

	1	2	3	4	5
1		0*	2*	4	3
2			6	2*	3
3				5	3
4					3

where $m=5$, $n=3$, $d=1$, $m_0=2$, and $r=2$. The minimum value listed is zero, which is then marked; we form the list

(6)

	3	4	5
1, 2	8	6	6
3		5	3*
4			3

The minimum unmarked value is 2 in (5); which is then marked; we form the list

(7)

	2	4	5
1, 3	8	11	8
2			5
4		4	5

The minimum unmarked value is 2 in (5); which is then marked; we form the list

(8)

1	1	3	5
1		4	5
2, 4	6	13	8
3			5

The solution is apparent in (6) for the element 3. The objective function value is 3 (or 6 in the original problem) where $\bar{T}_1 = (1, 2)$, $\bar{T}_2 = (3, 5)$, $\bar{T}_3 = 4$. Thus we have

(9)

$$X = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}.$$

Note an alternate solution in (6) with $\bar{T}_1 = (1, 2)$, $\bar{T}_2 = 3$, $\bar{T}_3 = (4, 5)$.

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INTERACTIONS BETWEEN THE EXPERIMENT DESIGNER AND THE COMPUTER*

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ABSTRACT

Popularized scientific accounts have publicized the use of the computer for solving combinatorial problems associated with the construction of specific kinds of experiment designs, such as sets of orthogonal Latin squares and Hadamard matrices. Historically, the regular combinatorial properties associated with such designs have been required to obviate extremely tedious calculations required for design analysis. Now computers are generally available for performing these computations, so that interest in regular designs is waning. The experiment designer is now free to select a design for its appropriateness to the experiment and its statistical properties. Once again, he interacts with the computer, using it to evaluate the statistical properties of designs, even before the experiment is carried out. Based on such calculations, he may make a selection from a number of ad hoc designs constructed for a given experiment without regard for combinatorial properties.

Although the experiment designer will always be a frequent computer user, the most important interaction between the two will probably be in the application of experiment design to computer simulation. Parametric studies of comprehensive system models that have been programmed for computers can be conducted efficiently and economically with experiment design and the associated analysis techniques. A typical example is given.

INTRODUCTION

The Nature of Factorial Designs

Past and future relationships between the experiment designer and the computer are explored in this paper. Although attention is focused on fractional factorial designs, other types of designs are similarly related to the computer. The purpose of an experiment may be viewed as the discovery of the effects of a number of independent variables or factors. Factorial experiments are characterized by the restriction of these factors to a small number of values or levels, usually from two to six. Each trial run must be conducted at a grid point, called a treatment combination, which is a specification of a particular level for each of the factors under investigation. The experiment design consisting of one run at each of the possible treatment combinations is called the full factorial design. If any treatment combination is omitted, the design is an incomplete factorial design. By convention, the levels of an s-level factor are assigned the integers 0, . . ., s-1. Any design can be specified by the design matrix D, whose columns correspond to the factors, and whose rows correspond to the treatment combinations included in the design. The entry in the *i-th* row and *j-th* column of matrix D is the integer associated with the level of the *j-th* factor in the *i-th* treatment combination which is included in the design.

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*The initial draft of this paper was written while the author was at Rocketdyne, a Division of North American Rockwell Corporation. The work was conducted under Independent Research and Development funding. The final version was completed at McDonnell Douglas Astronautics Company—Western Division.

Analysis of Factorial Designs

The quantitative analysis of factorial experiments is conducted by computing statistical estimates of a set of parameters used to describe the response. The model is set up such that the experimental outcome of a trial at any treatment combination is equal to a linear combination of these unknown parameters, perturbed by a random error. It can be written in matrix form as

$$Y = X\beta + e,$$

where Y is the vector of responses that are observed at the various treatment combinations, β is the vector of parameters, e is a vector of independent errors assumed to have common variance σ^2, and X is a coefficient matrix. To describe precisely the construction of the matrix X, it will be necessary to describe first the definition of β.

The parameters (elements of β) are conventionally the grand mean, main effects for each of the factors, and interaction effects involving pairs or triplets of the factors. They are defined in terms of the expected responses at the treatment combinations. The grand mean is the average of the expected responses, with the average taken over all treatment combinations of the full factorial (not just those included in a specific design). The main effects for a given factor at s levels are s-1 in number. Let $\mu_i (i = 0, 1, \ldots, s\text{-}1)$ be the average of the expected responses for all treatment combinations in which the given factor appears at level i. Then the s-1 main-effect parameters are defined by a set of s-1 independent contrasts on the μ_i. For example, a common convenient parameterization for a three-level factor consists of the two parameters $(\mu_2 - \mu_0)/2$ and $(\mu_2 - 2\mu_1 + \mu_0)/6$. Interactions between a pair of factors, one with s_i levels and the other with s_j levles, are defined by means of $\mu_{ij} (i = 0, \ldots, s_i\text{-}1; j = 0, \ldots, s_j\text{-}1)$, the average of the expected responses for all treatment combinations in which the first factor of the pair appears at level i and the second at level j. Three-factor interactions are defined by means of the quantities μ_{ijk}, which are defined analogously. Interactions between any number of factors can also be defined in this way.

Once the parameters are defined in terms of the expected responses at the treatment combinations, the process can be reversed, and the expected responses expressed in terms of the parameters. The factorial structure of an experiment can best be exploited in situations in which the expected responses can be expressed approximately as a function of only a small subset of all the parameters that can be defined. That is, the response is smooth enough so that many of the parameters, usually interaction parameters, can be assumed to be of negligible magnitude. The expectation of the response vector Y from a particular experiment can therefore be written as $EY = X\beta$, where β contains only those parameters that are not to be ignored. The rows of the coefficient matrix X correspond to the runs of the design that produced Y. The columns of X correspond to the parameters; the *j-th* column of X is called the coefficient vector for the *j-th* parameter. The coefficient vector for the grand mean has all its elements equal to 1. A coefficient vector for a main effect may contain only as many distinct entries as there are levels of the associated factor.

After an experiment is carried out and a particular realization of the response vector Y is observed, the parameter vector β can be estimated by least squares, which gives

$$\hat{\beta} = (X'X)^{-1}X'Y,$$

if the matrix $(X'X)$ is nonsingular. The variance of β is

$$\text{Var}\,(\hat{\beta}) = \sigma^2 (X'X)^{-1}.$$

Thus, the properties of the design, inasmuch as they are reflected in the variances and covariances of the elements of $\hat{\beta}$, are tied up in the matrix X. Therefore, these properties can be established before the experiment is actually carried out.

An important class of designs, called orthogonal designs, are defined by the condition that $X'X$ is a diagonal matrix. Orthogonal designs have until recently received most of the attention in the literature. Orthogonal designs are very easy to analyze because the matrix $X'X$ is so easy to invert. Within the class of designs for which the lengths of the coefficient vectors are fixed, an orthogonal design minimizes each of the variances of the components of $\hat{\beta}$, if such a design exists. There is a discussion of the appropriateness of this restriction in Ref. [26]. Under the parameterization which is usually chosen for factorial designs, the full factorial is an orthogonal design.

COMPUTERS FOR CONSTRUCTION OF DESIGNS

Early Construction Techniques

Much of the early work in the field of factorial design, conducted by Fisher and Yates, dealt with the problem of blocking a replicated full factorial design (Ref. [29]). Their designs were used for agricultural field trials; the factors were usually fertilizer type and amount, and the blocks were physical groupings of contiguous plots. They found that the effect of inhomogeneity of soil quality can be lessened by choosing a smaller block size, and that the treatment combinations can be assigned to the blocks in such a way that the presence of block effects does not disturb the main effects or interactions of primary interest. This condition holds true if the coefficient vectors of X for block effects are orthogonal to the remaining coefficient vectors, and it is achieved by the technique of confounding high-order interactions (arranging the blocks so that their coefficient vectors are in the subspace spanned by the coefficient vectors for certain interactions).

As the need for incomplete factorial designs became evident, methods for constructing designs were developed along the same lines that had yielded confounded designs. In fact, one block of a confounded design may often be used as a fractional factorial. More generally, the well-known class of fractional factorials can be constructed by use of simple properties of groups that are direct products of groups of small order (2 or 3). All the resulting designs are orthogonal, and even today are the most useful incomplete factorial designs. As more complicated designs have been required, it has been necessary to develop much more sophisticated machinery for generating new designs. In general, these methods have followed the old patterns in that they restrict attention to orthogonal designs.

The conditions under which a design is orthogonal can be reduced to combinatorial conditions quite easily. With the advent of high-speed electronic digital computers, it has become possible to solve these combinatorial problems with procedures that augment sophisticated theory by enumerative search processes (see Ref. [13] for an expository overview).

Construction of Latin Squares

Computing machinery has been used to find sets of mutually orthogonal Latin squares. This work most likely began almost immediately after the celebrated discovery of a set of mutually orthogonal Latin squares of side 22 by Bose and Shrikhande [5], thus negating a conjecture attributed to Euler [7]. A Latin square of side n is an $n \times n$ array of cells with each cell containing one of n distinct objects, typically the first n Latin letters. The letters are arranged in such a way that each row and column of the square contains each letter exactly once. A pair of orthogonal squares is such that each of the n letters of the first square coincides in row and column with the n letters of the other square exactly

once. Sometimes the same arrangement is achieved within a single square by filling the cells with a Latin and a Greek letter so that each Latin letter appears with each Greek letter exactly once (hence, the term Graeco-Latin square). Sets of more than two Latin squares are called mutually orthogonal if each pair is orthogonal. A set of r mutually orthogonal Latin squares of side n provides an experiment design for $r+2$ factors each at n levels. Euler's conjecture (1782) was that no pair of orthogonal squares exists if n is of the form $4k+2$. For $n=6$ the conjecture was proved true by Tarry [23] in 1901 by use of exhaustive enumeration. It was not until 1959 that the conjecture was disproved.

It is not feasible to use enumerative procedures to generate Latin squares directly. In order to make problems manageable on even the fastest computers, a device that exploits the structure of the problem must be used. Parker [20] divided the problem into two parts. First, starting from a given Latin square, the computer is used to obtain a set of transversals of the original square. A transversal is a set of n cells; one cell is contained in each row and column and each letter appears once in the transversal. Two transversals are illustrated in the Latin square in Figure 1. After a set of n transversals that are mutually exclusive has been constructed, a Latin square orthogonal to the first may be constructed by assigning a distinct letter to each of the transversals. This technique was used by Parker to derive sets of mutually orthogonal 10×10 Latin squares (Ref. [20]).

Other computer techniques for constructing sets of orthogonal Latin squares are given in Refs. [3, 4, 16], all of which are based on the concept of orthogonal mappings of a group. Given a group G of order n and a 1-to-1 mapping α of G into itself, it is possible to construct a Latin square by placing in the cell in the i-th row and j-th column the element $(\alpha i)j$. Two mappings, α and β, are defined to be orthogonal if the mapping $\gamma i=(\alpha i)(\beta i)^{-1}$ is a 1-to-1 mapping of G into itself. Latin squares corresponding to orthogonal mappings are orthogonal. The conceptual relationship between Latin squares and mappings was developed by Mann [18].

The technique used by Bose, et al. to construct a Latin square of order $4t$ was to consider the group of order $4t$ that is the direct product of the group of order 2 with the cyclic group of order $2t$. First, a Hadamard matrix was constructed of order $4t$. The first row of the matrix was completed by associating the elements of the group of order $2t$ with each of the 1's and each of the -1's in lexicographic order. The computer was used to enumerate all possible ways to adjoin the elements of the group of order $2t$

[a]	b	(c)	d	e	f	g	h	i	j
b	i	d	c	f	e	h	[g]	j	(a)
c	j	f	g	(d)	a	[i]	e	h	b
d	h	a	[j]	i	g	(b)	f	c	e
e	(g)	h	f	c	j	a	i	[b]	d
f	a	j	(e)	[h]	i	d	b	g	c
g	f	[e]	h	b	d	c	(j)	a	i
h	e	b	i	a	[c]	j	d	(f)	g
(i)	[d]	g	a	j	b	f	c	e	h
j	c	i	b	g	(h)	e	a	d	[f]

FIGURE 1.—A Latin square of side 10 with two transversals illustrated.

to the remaining rows of the Hadamard matrix so that the mapping corresponding to the rearrangement of the group of order $4t$ is orthogonal to the identity mapping provided by the first row. That is, the elements of the group of order $2t$ were adjoined to the 1's and to the -1's to make up all the elements of the group of order $4t$. Let S_i be the set of mappings constructed for the i-th row of the Hadamard matrix. The computer was then used to search the sets S_2 and S_3 for all possible pairs of orthogonal mappings. Then the mappings in S_4 were tested by the computer to determine orthogonality with the selected pairs.

This process was continued as long as possible and, in fact, produced two sets of five mutually orthogonal squares of side 12. Similar techniques were used by Johnson, et al. [16], except that the orthogonal mappings were constructed by a different technique. Bose, et al. [4] extended their work to include double and triple mappings, which can be used in a similar fashion.

Construction of Hadamard Matrices

Another application involving computer solution to experiment design problems is the construction of Hadamard matrices. A Hadamard matrix is an $n \times n$ matrix each of whose elements is -1 or 1 and such that the rows of the matrix are orthogonal to each other (using the ordinary vector inner product). In its original form as proposed by Hadamard [12], the problem was to construct matrices whose elements are no greater than 1 in modulus and whose determinant is maximized. The equivalent form of the problem stated here is more natural in the context of experiment design. Such a matrix provides an orthogonal experiment design for up to $n-1$ two-level factors in exactly n runs. Paley [19] gives several ways of constructing such matrices based on the properties of those elements of a Galois field that are perfect squares (called quadratic residues). Paley's techniques provide Hadamard matrices of size $4k$ for all integers k up to 25, except for $4k = 92$. It was of interest to determine if in fact a Hadamard matrix of this order existed.

Golomb and Baumert [11] demonstrated the existence of a Hadamard matrix of order 92 by constructing such a matrix. The technique they used was to construct four symmetric matrices, A, B, C, and D, such that $A^2+B^2+C^2+D^2=92I$ and such that each of the matrices is a cyclic permutation of its first row. The desired Hadamard matrix is then of the form

$$\begin{bmatrix} A & B & C & D \\ B & -A & D & -C \\ C & -D & -A & B \\ D & C & -B & -A \end{bmatrix}$$

The four matrices were found by means of a systematic search procedure called the backtrack technique (Ref. [10]). The solution was obtained in less than an hour of machine time on the IBM 7094.

Construction by Means of Linear Programming

An alternative technique for constructing incomplete factorial designs with given properties was considered by this author. Such properties as orthogonality of the entire design or of particular coefficient vectors for main effects and interactions can be reduced to a matrix of linear constraints that must be satisfied by the incidence vector of the design. This vector is defined as the number of occurrences of each of the possible treatment combinations in the given design. Ordinarily for a real design, the elements of this vector will be either 0 or 1. There are for this type of problem fewer constraints than variables. The problem of constructing a design with given properties and with a minimum number

of runs can therefore be formulated as a problem of obtaining the solution vector that satisfies the constraint matrix, whose elements are positive integers, and the sum of whose elements is minimum. This is a linear programming formulation.

Although this fact appears to be of some theoretical interest, it does not by itself provide a feasible technique for actually constructing designs. Numerous attempts were made by the author to derive both known and new designs using linear programming. Success was achieved only when the number of possible treatment combinations was small; for example, the 2^5 design in 16 runs, itself a Hadamard matrix, was derived. The problem one encounters is the extreme degeneracy of the solution vectors of interest. This means that the number of constraints is greater than the number of nonzero components of the solution vector. This fact causes the linear programming algorithm to churn away for many (perhaps several hundred) iterations before a change is observed in the solution vector. When such a change does occur, it usually produces a solution that is nonintegral. After subsequent changes, however, an integral solution is often regained.

Even though the linear programming approach has not proved successful, it is possible that the information contained in the constraint matrix for such designs can be used with a search technique, such as the backtrack technique, in order to obtain solution vectors that represent new and potentially useful designs, but this has not as yet been tried. A summary of results obtained using this approach is contained in Ref. [25].

COMPUTERS FOR ANALYSIS OF DESIGNS

The Use of Orthogonal Designs

Originally, a compelling reason for using an orthogonal design was to simplify the analysis of resulting data; for if the matrix $X'X$ is not diagonal, its inversion involves extensive tedious computation. Unless experimental units were extremely expensive, the cost of analyzing a nonorthogonal design did not make it economically attractive to run such a design. Therefore, much research effort was expended on finding orthogonal designs and designs with special patterns to the $X'X$ matrix which allowed them to be inverted relatively easily. Examples of the latter types are found in such designs as three-eighth replicates or, more generally, the irregular factorial designs of Addelman [1], John [15], and others.

With the advent of high-speed digital computers, the cost of inverting large matrices reliably and accurately has dropped to a negligible amount. Therefore, there is no particular need for orthogonal designs from the analysis viewpoint. There are, however, other reasons for preferring orthogonal designs if they are convenient for a particular experiment.

One highly touted aspect is the optimality properties which these designs enjoy. Another is that it is easy for the uninitiated experimenter to visualize the structure of the data from an orthogonal design. The raw averages of the responses at each of the levels of the factors provide the estimates of the effects of those factors without bias. In nonorthogonal designs, however, the estimates of the effects are more complicated linear combinations of the responses. The expected value of a simple average of the responses for one level of a factor will involve other factors. Thus, the results of the experiment are harder to interpret and the results of the analysis harder to accept, since they cannot be seen transparently in the raw data.

It is clear, therefore, that whenever convenient one should choose an orthogonal design. There are many situations, however, in which running an orthogonal design entails severe compromises.

These compromises are as follows:

1. Restriction of the number of factors included in the design.
2. A lack of freedom in the number of levels for the factors included.
3. A severely large requirement on the number of runs.

The number of runs in the smallest orthogonal design will often be much larger than the number of experiments that can be afforded. In such situations there is no longer any rational objection to the deliberate use of nonorthogonal designs.

The Choice and Evaluation of Designs

If nonorthogonal designs are to be used, it is appropriate to develop objective criteria for their selection. Some criteria have been proposed that are based on combinatorial properties of the designs. It is more reasonable, however, to use essentially the same criteria that lead to the preference for orthogonal designs, if they are available. Various criteria for optimality of designs have been proposed by many authors. Kiefer [17] has studied a number of these in detail from the theoretical viewpoint. These criteria include the minimization of the determinant of the matrix $(X'X)^{-1}$ (called the D optimality criterion by Kiefer), minimization of the average of the variances of the components of β (A optimality), minimization of the maximum eigenvalue of $(X'X)^{-1}$ (E optimality), minimization of the maximum variance of a fitted response over an appropriately selected experimental region (G optimality), and minimization of the variance of a fitted value integrated over the experimental region (Ref. [17]). Box and Draper [6] have considered the integrated mean square error over the experimental region when the true model is of a higher order than the one for which the fit is obtained. For factorial designs I prefer the variance of a fitted value averaged over all the points of the full factorial design. This criterion is conveniently converted into an index of merit with value unity for orthogonal designs (Ref. [28]). The selection of a nonorthogonal design can be based on any of these criteria. It will be noted that designs may be evaluated before the experiment is run from the properties of the design matrix and do not involve the data from the experiment.

It is very difficult to anticipate all the possible constraints that may be imposed upon how an experiment is to be conducted. For example, certain portions of the experimental region may be excluded, an unusual set of interactions may be required, or factors may be naturally at different numbers of levels. This means that it is not feasible for an experiment designer to anticipate all possible experiments. An attractive alternative followed by this author is to construct a number of candidate designs for a given experiment and to evaluate them using a convenient computer program which was developed for this purpose. A number of techniques for constructing candidates can be obtained from the literature on nonorthogonal designs, which is discussed in Ref. [24]. A type of design that has received much attention is the irregular fraction, obtained by adjoining several regular fractions (Refs. [1] and [15] have examples). Many good designs have the property of permutation invariance, which means that the X matrix remains unaltered under permutations of those factors that appear at the same number of levels. Several classes of designs have been derived by use of correspondence techniques, such as collapsing the levels of a three-level factor to obtain a two-level factor. This technique was used by Addelman [2] to obtain proportional frequency designs. Often, however, a good design can be constructed by a cut-and-try process. After a number of designs have been constructed, they can be evaluated easily by use of the computer routine described in Ref. [28], which computes the variances of the estimates and several indexes of merit, including the one mentioned above.

Recently, this approach has gained increasing value as a method for obtaining sets of designs. The author has recently published a catalog of designs for two- and three-level factorial experiments containing 20 or fewer runs (Ref. [27]). The same program was used by Rechtschaffner to evaluate the designs contained in Ref. [21].

EXPERIMENT DESIGNS FOR COMPUTERS

Experiments Conducted Within the Computer

Computer programs of comprehensive mathematical models for the behavior of complex systems are becoming more widely used in practically every field of science. Typical examples are simulation programs for design studies on water resources systems (Ref. [14]), a program for studying the mechanisms of biological reproduction at the molecular level (Ref. [30]), a program for studying maintenance policies for military bases (Ref. [9]), a program for analyzing heating of infiltrated porous tungsten (Ref. [8]), and a program for simulating global weather patterns (Ref. [22]). Such computer programs are often used to study the behavior of the system as the value of input parameters are changed; this technique substitutes numerical experimentation on the computer for physical experimentation on the system itself. The computer time involved in exercising the computerized model for a particular combination of values of input parameters and the number of possible combinations of values of input parameters can make it extremely desirable to run only those combinations that comprise a factorial experiment design.

The purpose of an experimental investigation of a computerized model is to find a simple but accurate graduating function that will represent over a region of interest the true function which comprises the computer model itself. The usual definition of experimental error does not apply, since a repeat of a test case run under identical conditions is bound to give identical answers. Because there is no such thing as experimental error in the usual sense, there is no basis on which to test statistical hypotheses. In such situations, highly saturated nonorthogonal designs may be desirable. The criterion by which the value of a parameter is judged significant is whether or not changing it over its range produces an effect on the response that is important on physical grounds. The designs contained in Ref. [27], which were evaluated by a computer, are good designs for parametric studies conducted on a computer, because they do not contain extra observations to provide for an independent error estimate.

An Example of a Parametric Study

One such parametric study, for which the data were readily available, was conducted upon a computer program for predicting the performance of a small liquid rocket engine. The experimental variables were the contraction ratio or ratio of the conbustion chamber diameter to the diameter of the throat of the rocket nozzle (three levels), the length of the combustion chamber (three levels), the shape of the combustion chamber (whether cylindrical or conical) (two levels), and the velocity with which the oxidizer and fuel are injected into the combustion chamber (two levels). These four variables will be designated by the letters A, B, C, and D, respectively. Over the region covered by the $3^2 \times 2^2 = 36$ possible treatment combinations, it will be assumed that the response can be adequately fit by the following model:

$$Y = \text{const} + A_L + B_L + C_L + D_L + A_Q + B_Q + A_L B_L + A_L C_L + A_L D_L + B_L C_L + B_L D_L + C_L D_L.$$

Here A_L is the linear effect of factor A, which is defined as one-half the difference between the average response at the highest level of the factor and the average response at the lowest level; A_Q, the quadratic effect, is one-sixth the average response at the low level, minus one-third the average response at the middle level, plus one-sixth the average response at the high level; other terms are defined analogously, the product terms being interactions. A design is given in the catalog (Ref. [27]) which permits these effects to be estimated from 14 runs. The treatment combinations are given in Table 1 along with

TABLE 1
A 14-Run Experiment Conducted on a Computer

Treatment combination	Response
0000	1.93
0011	1.63
0101	2.20
0201	2.55
0210	2.46
1001	2.06
1110	2.57
1111	2.34
1200	3.50
2000	2.75
2011	2.01
2100	3.86
2201	3.67
2210	3.80

the value of a rocket chamber performance index obtained with the computer program for each treatment combination. The matrix X for this design is as follows:

$$X=\begin{bmatrix}
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1\\
1 & -1 & 0 & -1 & 1 & 1 & -2 & 0 & 1 & -1 & 0 & 0 & -1\\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1\\
1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\
1 & 0 & -1 & -1 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & -1 & -1\\
1 & 0 & 0 & 1 & -1 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & -1\\
1 & 0 & 0 & 1 & 1 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & -1 & -1 & -2 & 1 & 0 & 0 & 0 & -1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1\\
1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1\\
1 & 1 & 0 & -1 & -1 & 1 & -2 & 0 & -1 & -1 & 0 & 0 & 1\\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1\\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1
\end{bmatrix}$$

A least-squares computer program was used to obtain the fitted model:

$$Y=2.62+0.48A_L+0.55B_L-0.20C_L-0.16D_L+0.03A_Q-0.05B_Q+0.16A_LB_L$$
$$-0.05A_LC_L-0.08A_LD_L-0.01B_LC_L-0.07B_LD_L+0.04C_LD_L.$$

The residual standard deviation from this design was 0.18, based on 1 degree of freedom. In point of fact, all 36 treatment combinations were actually run. The fitted values obtained with the above model were compared with the observations; the standard deviation of the differences is 0.20, based on 20 degrees of freedom. A model containing more terms and based on all 36 observations was also used to obtain fitted values. The residuals were on the average only slightly smaller. Substantially the same conclusions would be drawn from the 14-run design (Table 1) as from the entire 36-run design.

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DIFFERENTIAL GAME THEORETIC ANALYSIS OF A PROBLEM OF WARFARE

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ABSTRACT

The present article considers a problem of missile warfare. It is formulated as a problem of economics and treated as a differential game. Some optimal strategies of targeting and firing of missiles are derived. Conditions under which such a war can be prevented are also developed.

1. INTRODUCTION

A problem of missile warfare was introduced as an example in a previous article ref. [1]. It is considered here in a more general form. Two basic assumptions are made to develop a mathematical model. These are, (a) the countervalue targets of both participants in the war have strong defenses (e.g., ABM defense networks), and (b) the missile stocks of both participants are approximately equal. Under these assumptions, the problem becomes one of allocating scarce resources, over a period of time, to achieve some optimal results. In ref. [2], the problem of optimal targeting and firing of missiles has been considered for the following cases: (a) where the strategy of the opponent *is known*, (b) where the opponent's strategy is to be influenced, and (c) at termination of war. The mathematical model of the problem developed in [2] is used here. It is assumed, however, that the opponent's strategy is *not known a priori*. The problem is treated as a differential game, and the optimal strategies of both participants in the war are developed. Also, some conditions of deterrence are developed under which such wars can be prevented.

A word of caution is perhaps necessary about the nature of the results developed here. These results are influenced by the following factors: (a) the mathematical model, (b) the performance index, (c) the method of analysis, and (d) the emphasis on obtaining analytical results. Some of the results may appear to be out-dated. The important point, however, is that a quantitative analysis of the problem of missile warfare can be made to provide plausible results. Perhaps, a more sophisticated model with computer aided analysis will yield more satisfactory results. It is the author's belief, however, that a preliminary analysis is necessary to partially justify the use of a computer in a detailed analysis.

2. DYNAMIC MODEL OF A MISSILE WAR

Let X and Y denote the opponents participating in a missile war. At time t, let $x_1(t)$ and $y_1(t)$ denote the number of missiles, and $x_2(t)$ and $y_2(t)$ denote the measure of destruction of civilian targets of participants X and Y, respectively. Such a measure of destruction can be a cumulative total damage of countervalue targets.

Let $\alpha(t)$ and $\beta(t)$ denote the rates of fire of missiles, and $\alpha'(t)$ and $\beta'(t)$ denote the counterforce proportions of missiles of X and Y, respectively. The range of values of $\alpha(t)$ and $\beta(t)$ are assumed

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to be constrained as follows: $0 \leq \alpha(t) \leq a$ and $0 \leq \beta(t) \leq b$, where $0 < a, b < 1$. Simultaneous firing of all missiles is assumed to be impossible since one launching device may have several missiles to fire. The counterforce proportions $\alpha'(t)$ and $\beta'(t)$ can take values between 1 (pure counterforce), and 0 (pure countervalue) when there is no collocation of targets. Let f_x, f_y and v_x, v_y denote the counterforce and countervalue effectiveness, respectively, for X and Y. At time, t, by definition $f_x(t)$ = number of missiles of Y destroyed per counterforce missile of X, and $v_x(t)$ = amount of destruction of civilian targets of Y per countervalue missile of X. Similar definitions can be given for $f_y(t)$ and $v_y(t)$. It is assumed that the parameters f_x, f_y, v_x, and v_y can be approximated by real numbers from the interval [0, 1]. Such approximations are justified in view of the two basic assumptions introduced in Section 1. The dynamic equations can now be written as follows:

(1a) $$\frac{dx_1}{dt} = -\alpha x_1 - \beta\beta' f_y y_1,\ x_1(t_o) = x_1^o,$$

(1b) $$\frac{dx_2}{dt} = (1-\beta')\beta v_y y_1,\ x_2(t_o) = 0,$$

(2a) $$\frac{dy_1}{dt} = -\beta y_1 - \alpha\alpha' f_x x_1,\ y_1(t_o) = y_1^o,$$

(2b) $$\frac{dy_2}{dt} = (1-\alpha')\alpha v_x x_1,\ y_2(t_o) = 0.$$

3. MISSILE WAR AS A DIFFERENTIAL GAME

This section is concerned with conditions of termination and performance index of a missile war. Conditions of termination can be specified by means of upper bounds on acceptable damages to countervalue targets. The theory of differential games, however, can not yet be applied to such models to provide tangible results. Hence, the duration of war $[t_o, T]$ is assumed to be preassigned. This restrictive assumption will be removed in a future paper.

The duration of war $[t_o, T]$ cannot be chosen arbitrarily. From physical considerations

$$x_1(t) \geq 0,\ y_1(t) \geq 0, \text{ and } t \in [t_o, T].$$

If T is chosen arbitrarily, however, the optimum solutions of Eqs. (1a) and/or (2a) may become negative. Since missiles are considered scarce resources in this model, the duration of war is determined by counterforce considerations.

Choice of T depends on the allowed ranges of $\alpha(t)$ and $\beta(t)$ and estimated values of the parameters involved. For example, suppose $f_x = 1$, $f_y = 0$, $\alpha' = \beta' = 1$, and $\alpha = \beta = b (a = b)$. This will produce a severe drain on the missile stock of Y. In this case $x_1(t) > 0$, $0 \leq t < \infty (t_o = 0)$ and $y_1(T) = 0$ where $T = y_1^o / x_1^o b$. As another example, suppose $f_x = f_y = 1$, $\alpha' = \beta' = 1$ and $\alpha = \beta = b (a = b)$. Let $x_1^o - y_1^o = m > 0$ and $x_1^o + y_1^o = n$. Then $x_1(t) > 0$, $0 < t \leq \infty (t_o = 0)$, and $y_1(T) = 0$, where $T = \frac{1}{2b} \ln (n/m)$ and $T \to \infty$ as $m \to 0$. In other words, once the values of the parameters are estimated, a suitable T can be chosen by considering the worst rate of loss of missiles.

Let the performance index of the war be defined as follows:

$$J=\frac{K_1}{2}\left[\frac{y_1(T)}{x_1(T)}\right]^2+\frac{K_2}{2}\left[x_2(T)^2-y_2(T)^2\right]. \tag{3}$$

where X is trying to minimize J and Y is trying to maximize it.

The assumption regarding the duration of the war implies that the sets of pure strategies for X and Y are nonempty. It is also assumed that these sets contain the optimal strategies of X and Y. Thus, the theory of differential games as presented by the author in [1] can be applied to this problem. The Eqs. (1a), (1b), (2a), and (2b) will be referred to as the equations of state of the differential game. The initial conditions then become the state of the game at time t_o.

4. OPTIMAL STRATEGIES

Let H denote the Hamiltonian of this differential game. Then

$$\begin{aligned} H&=p_1(-\alpha x_1-\beta\beta' f_y y_1)+p_2(1-\beta')\beta v_y y_1+\lambda_1(-\beta y_1-\alpha\alpha' f_x x_1)+\lambda_2(1-a')\alpha v_x x_1 \\ &=[(\lambda_2 v_x-p_1)-(\lambda_1 f_x+\lambda_2 v_x)\alpha']\alpha x_1+[(p_2 v_y-\lambda_1)-(p_1 f_y+p_2 v_y)\beta']\beta y_1, \end{aligned} \tag{4}$$

where $p_1(t)$, $p_2(t)$, $\lambda_1(t)$, and $\lambda_2(t)$ are the usual adjoint variables. Thus the adjoint differential equations for the differential game can be written as follows:

$$\frac{dp_1}{dt}=[(\lambda_1 f_x+\lambda_2 v_x)\alpha'-(\lambda_2 v_x-p_1)]\alpha, \qquad p_1(T)=-K_1\frac{y_1(T)^2}{x_1(T)^3} \tag{5a}$$

$$\frac{dp_2}{dt}=0 \qquad , \qquad p_2(T)=K_2 x_2(T) \tag{5b}$$

$$\frac{d\lambda_1}{dt}=[(p_1 f_y+p_2 v_y)\beta'-(p_2 v_y-\lambda_1)]\beta. \qquad \lambda_1(T)=K_1\frac{y_1(T)}{x_1(T)^2} \tag{6a}$$

$$\frac{d\lambda_2}{dt}=0 \qquad , \qquad \lambda_2(T)=-K_2 y_2(T). \tag{6b}$$

From physical considerations $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ are all positive quantities and hence $\alpha(t)x_1(t)\geq 0$ and $\beta(t)y_1(t)\geq 0$. It is evident from Eq. (4), that the optimal choice of targeting strategies $[\alpha'(t), \beta'(t)]$ precedes and influences the optimal choice of firing strategies $[\alpha(t), \beta(t)]$.

Optimal Targeting Strategies:

For X:	For Y:
$\lambda_1(t)f_x+\lambda_2(t)v_x>0,\ \alpha'(t)=1$	$p_1(t)f_y+p_2(t)v_y>0,\ \beta'(t)=0$
$\lambda_1(t)f_x+\lambda_2(t)v_x<0,\ \alpha'(t)=0.$	$p_1(t)f_y+p_2(t)v_y<0,\ \beta'(t)=1.$

Optimal Firing Strategies:

For X:

$$\lambda_2(t)v_x-[\lambda_1(t)f_x+\lambda_2(t)v_x]\alpha'(t)>p_1(t),\ \alpha(t)=0$$
$$\lambda_2(t)v_x-[\lambda_1(t)f_x+\lambda_2(t)v_x]\alpha'(t)<p_1(t),\ \alpha(t)=a.$$

For Y:

$$p_2(t)v_y - [p_1(t)f_y + p_2(t)v_y]\beta'(t) > \lambda_1(t),\ \beta(t) = b$$
$$p_2(t)v_y - [p_1(t)f_y + p_2(t)v_y]\beta'(t) < \lambda_1(t),\ \beta(t) = 0.$$

Note that along the optimal trajectories of the differential game $dp_1/dt \geq 0$ and $d\lambda_1/dt \leq 0$. Also $p_1(T) \leq 0$ and $\lambda_1(T) \geq 0$. Hence, along the optimal trajectories $p_1(t) \leq 0$, $\lambda_1(t) \geq 0$, $t \in [t_0, T]$. Also $\lambda_2(t) \leq 0$ and $p_2(t) \geq 0$ are constants for $t \in [t_0, T]$. Suppose $\alpha'(T) = 1$, i.e., $\lambda_1(T)f_x > -\lambda_2(T)v_x$. Since $d\lambda_1/dt \leq 0$, $\lambda_1(t) \geq \lambda_1(T)$ $t \in [t_0, T]$. Hence, $\lambda_1(t)f_x > -\lambda_2(T)v_x$ and $\alpha'(t) = 1$, $t \in [t_0, T]$. Thus, if the optimal terminal targeting strategy is pure counterforce then it must be pure counterforce over the duration of the war. On the other hand, if $\alpha'(T) = 0$ then there may be a switching to $\alpha'(t) = 1$ for some $t \in [t_0, T]$. Hence, optimal targeting strategy always changes from pure counterforce to pure countervalue and never in the opposite direction. The decision rules for optimal targeting and firing of missiles can be given physical interpretations in terms of the concept of "shadow prices" of economic theory (see [2]). At time t, let $\lambda_1(t)$ denote the value to X of destroying a missile of Y, and $-\lambda_2(t)$ denote the value to X of destroying a civilian target of Y. Then for X, the decision rule of targeting can be re-written as follows:

[(value to X of destroying a missile of Y) $\times$ (Counterforce effectiveness of X)] >

[(value to X of destroying a civilian target of Y) $\times$ (Countervalue effectiveness of X)]

implying $\alpha'(t) = 1$ (pure counterforce). Similar interpretations can be given to $p_1(t)$, $p_2(t)$, and the decision rule for optimal targeting of Y.

Now consider the decision rule for optimal firing of missiles of X. The quantity

$$[\lambda_1(t)f_x + \lambda_2(t)v_x]\alpha'(t) - \lambda_2(t)v_x$$

is the maximum value to X of firing a missile at time t. This equals $\lambda_1(t)f_x$ when $\alpha'(t) = 1$ (pure counterforce) and $-\lambda_2(t)v_x$ when $\alpha'(t) = 0$ (pure countervalue). $p_1(t)$ denotes the value to Y of destroying a missile of X at time t. Hence $-p_1(t)$ denotes the value to X of losing a missile at time t. Thus for X, $\alpha(t) = a$ whenever the maximum value of firing a missile is greater than that of losing it. A similar interpretation can be given to the decision rule for optimal firing of missiles of Y.

5. REGION OF DETERRENCE

A subset in the state space (R^4) of the differential game is defined to be a region of deterrence if, for every initial state of the game lying in this subset, the optimal firing strategies are $\alpha(t) = \beta(t) = 0$, $t \in [t_0, T]$. Such a region of deterrence will be constructed in this section under the assumption that *both X and Y are using optimal strategies.* Note that since $\alpha(t) = \beta(t) = 0$, $t \in [t_0, T]$, the targeting strategies are of no significance during $[t_0, T]$; however, the deterrent region depends on the targeting strategies.

The following procedure is used to construct the deterrent region. The terminal state of the game is chosen arbitrarily. This in turn determines the optimal terminal strategies. The adjoint differential equations are then integrated in reversed time to determine the switching times for the strategies. Then initial states of the game are sought to satisfy the terminal condition. It is assumed that $K_1 > 0$.

CASE I: Suppose the terminal state of the game is such that $\alpha'(T) = \beta'(T) = 1$ and $\alpha(T) = \beta(T) = 0$. Then the right-hand sides of Eqs. (5a) and (6a) become zero. Hence, by solving the adjoint equations in reversed time it is clear that the adjoint variables are all constants, and no switchings of strategies take place. Hence $\alpha'(t) = \beta'(t) = 1$, and $\alpha(t) = \beta(t) = 0$, $t \in [t_0, T]$. Then from the equations of state

of the differential game it is evident that $x_1^o = x_1(T)$, $y_1^o = y_1(T)$, and $x_2(t_o) = y_2(t_o) = x_2(T) = y_2(T) = 0$. Now consider the decision rules for the optimal choice of $\alpha(t)$ and $\beta(t)$. Since $\alpha'(T) = \beta'(T) = 1$, these are:

For X:	For Y:
$p_1(T) + \lambda_1(T) f_x < 0$	$\lambda_1(T) + p_1(T) f_y > 0$

In terms of the initial state of the game, these can be rewritten as

(7a) For X: $y_1^o - f_x x_1^o > 0$

(7b) For Y: $y_1^o - \frac{1}{f_y} x_1^o < 0.$

The decision rules for the optimal choice of $\alpha'(T)$ and $\beta'(T)$ in this case are $\lambda_1(T) f_x > 0$ and $p_1(T) f_y < 0$, respectively. These are satisfied by any initial state of the game satisfying inequalities (7a) and (7b). Hence, in this case inequalities (7a) and (7b) together defines the region of deterrence.

CASE II: Suppose the terminal state of the game is such that $\alpha'(T) = 1$, $\beta'(T) = 0$, and $\alpha(T) = \beta(T) = 0$. Once again, no switching occurs for either $[\alpha'(t), \beta'(t)]$ or $[\alpha(t), \beta(t)]$. Now consider the expression $p_1(T) f_y + p_2(T) v_y$. Noting that $p_2(T) = K_2 x_2(T)$ and $x_2(T) = x_2(t_o) = 0$ it is clear that

$$p_1(T) f_y + p_2(T) v_y = p_1(T) f_y = -K_1 f_y \frac{y_1(T)^2}{x_1(T)^3} < 0.$$

But this contradicts the assumption that $\beta'(T) = 0$. Hence, no terminal state of the game exists such that $\alpha'(T) = 1$, $\beta'(T) = 0$, and $\alpha(T) = \beta(T) = 0$.

Similarly it can be shown that the following cases, (*i*) $\alpha'(T) = 0$, $\beta'(T) = 1$, and $\alpha(T) = \beta(T) = 0$, and (ii) $\alpha'(T) = \beta'(T) = 0$ and $\alpha(T) = \beta(T) = 0$ are not admissible. Hence the only deterrent region is given by the inequalities (7a) and (7b).

It has been shown in [1] that whenever the opponents are using pure counterforce targeting (not necessarily the optimal targeting strategy), the war can be prevented if the initial conditions satisfy inequalities (7a) and (7b). The region of deterrence decreases with increases in f_x and/or f_y. The region of deterrence is unbounded which opens the possibility of a spiralling arms race.

6. EXAMPLES

The method described in the previous section for the construction of the region of deterrence, can be used in general to obtain the optimal strategies of X and Y. Since there are four independent terminal variables $[\alpha'(T), \beta'(T), \alpha(T), \beta(T)]$ and each of them can take any of two possible values, there are 16 possible situations to be studied. From considerations of symmetry, the number of possible situations can be reduced to 10. Some of these situations are not admissible as has been shown in the previous section. A quantitative analysis of the war can be done by extensive simulation on a computer. Such results will be published in the future. Two special cases will be considered here for an illustrative purpose.

EXAMPLE 1: Suppose $K_1 > 0$ and $\alpha'(T) = \beta'(T) = 1$ (pure counterforce). From previous discussion on optimal targeting strategies (see section 4), it is clear that $\alpha'(t) = \beta'(t) = 1$, $t \in [t_0, T]$. The switching lines $p_1(T) + \lambda_1(T) f_x = 0$ and $\lambda_1(T) + p_1(T) f_y = 0$ can be used in the (λ_1, p_1) plane to determine the optimal terminal firing strategies of $\alpha(T)$ and $\beta(T)$. Since from physical considerations

$\lambda_1(T) \geq 0$ and $p_1(T) \leq 0$, it can be easily shown that $\alpha(T) = a$, $\beta(T) = b$ is not admissible. The possible optimal choice of $\alpha(T) = \beta(T) = 0$ leads to the deterrent conditions discussed previously. So let $\alpha(T) = a$ and $\beta(T) = 0$.

Let τ_α denote the switching time when $\alpha(\tau)$ switches to 0. Then $e^{-\alpha\tau_\alpha}[p_1(T) + \lambda_1(T)f_x] = 0$. Since by assumption $\alpha(T) = a$, $p_1(T) + \lambda_1(T)f_x > 0$. So $\tau_\alpha = \infty$ is the only solution. Hence no switching takes place for $\alpha(t)$ in $[t_0, T]$. Similarly let τ_β denote the switching time when $\beta(\tau)$ switches to b. Then

$$e^{-\alpha\tau_\beta} = \left[f_x - \frac{1}{f_y}\Big/ \dot{f}_x + \frac{p_1(T)}{\lambda_1(T)}\right] \leq 0.$$

Hence $\tau_\beta = \infty$ is the only possible solution and no switching takes place for $\beta(t)$ in $[t_0, T]$. Thus $\alpha(t) = a$, $\beta(t) = 0$, and $t \in [t_0, T]$. The initial conditions, for which the above optimal strategies are valid, satisfy the following inequality $y_1^0 - f_x x_1^0 < 0$. The case of $\alpha(T) = 0$ and $\beta(T) = b$ can be treated in a similar manner.

EXAMPLE 2: Suppose $K_1 = 0$. It is clear from Eqs. (5a) and (6a) that $p_1(T) = \lambda_1(T) = 0$. Consider $\lambda_1(T)f_x + \lambda_2(T)v_x = \lambda_2(T)v_x < 0$. Hence, $\alpha'(T) = 0$. Similarly $\beta'(T) = 0$. Also $\lambda_2(T)v_x - p_1(T) = \lambda_2(T)v_x < 0$. Hence, $\alpha(T) = a$. Similarly $\beta(T) = b$. Hence, in this case the optimal terminal targeting strategies are pure countervalue. The optimal terminal firing rates are maximum.

Suppose $\beta'(t = 0, t \in [T - \Delta, T]$, $\Delta \geq 0$. The equation of $\lambda_1(\tau)$, in reversed time, over $[0, \Delta]$ becomes $\lambda_1(\tau) = p_2(T)v_y[1 - e^{-b\tau}]$. Thus, $\lambda_1(\tau) < p_2(T)v_y$ for all $\tau \in [0, \Delta]$. Hence, $\beta(\tau) = b$ over $[0, \Delta]$. Thus, during the later phase of the war, optimal firing rates are maximum as long as the optimal targeting strategies are pure countervalue.

However, during the initial phase of the war, the optimal targeting strategies can be pure counterforce. In that case, to match the terminal condition, there will be at least one switch in $[t_0, T]$ for each optimal targeting strategy. Let $t_1 = T - \tau_1$ denote the last instant of switching when $\alpha'(t)$ switches to pure countervalue. The effects of the parameters of the system on t_1 are given below.

At τ_1, the decision rule for optimal choice of $\alpha(\tau)$ implies

$$[1 - e^{-b\tau_1}] = \frac{v_x}{v_y f_x}\sigma,$$

where

$$\sigma = -\frac{\lambda_2(T)}{p_2(T)} \geq 0.$$

The following inferences can be drawn from this relation.

All other parameters remaining constant:

(i) If f_x is small, $\alpha'(t)$ changes from 1 to 0 in the initial phase of the war. Since f_x is small, X should concentrate longer on countervalue targets than on counterforce ones.

(ii) If v_x is small, $\alpha'(t)$ changes from 1 to 0 at a later phase of the war. Since v_x is small, X cannot inflict much countervalue damage on Y. Hence, X should concentrate longer on destroying the missiles of Y in order to keep his own countervalue damage low.

(iii) If v_y is large, $\alpha'(t)$ changes from 1 to 0 at a later phase of the war. Since v_y is large, X should concentrate longer on destroying the attacking missile force of Y than on the countervalue targets. The effects of the ratios (v_x/f_x) and (v_x/v_y) can also be ascertained in a similar way.

7. CONCLUSIONS

The following conclusions about the problem of missile war can be drawn from the previous analysis.

(i) Optimal choice of targeting strategies precedes and influences the optimal choice of firing strategies. Optimal targeting strategy is either pure counterforce or pure countervalue and never mixed.

For $K_1 > 0$, it changes from pure counterforce to pure countervalue and never in the opposite direction. Thus, the initial targets are the missiles of the opponent. Then, during the course of the war, the targeting strategy switches to the civilian targets of the opponent and remains so for the rest of the war.

For $K_1 = 0$, the optimal targeting strategy can be pure countervalue over the entire duration of the war (e.g., if v_x/f_x is large and $T - t_0$ sufficiently small). It can be said in general that optimal strategy for targeting is counterforce followed by countervalue. It is to be noted that the duration of either one of the two targeting strategies can be zero. This same pattern for targeting strategy has been suggested in the past by many strategists.

(ii) Optimal firing strategy is either the maximum available rate or zero. This does not necessarily imply a massive retaliation. For example, note that an optimal strategy in a specific case can be of the following nature. Counterforce targeting with maximum rate of fire followed by a countervalue targeting with zero rate of fire. This is equivalent to reducing the opponent's military capability and withholding certain portions of the attack force as a countervalue threat.

If the war has a countervalue objective ($K_1 = 0$), however, it is possible for the optimal strategy to be countervalue with maximum rate of fire for an appreciable length of time.

The strategy of counterforce attack with a residual countervalue threat (which can be carried out if necessary) offers a reasonable way of avoiding a nuclear holocaust. Hence, it tends to increase the credibility of retaliation. On the other hand, countervalue strategy with maximum rate of fire tends to discourage one's opponent from taking undesirable actions.

(iii) The discussion in Section 5 and Example 2 makes it clear that a missile war can be deterred only when both participants have some counterforce objectives. In this case it is possible to have either a force stabilization, or a spiralling increase or *decrease* of arms. The controlling mechanism is of a political nature and hence, falls outside the scope of the present model.

In conclusion, note that the optimality of the strategies depend on the choice of the performance index. If one participant does not use the optimal strategies, he will be worse off in relation to the chosen performance index; however, he may be better off in his own way.

ACKNOWLEDGMENTS

This work was supported by the United States Air Force under the following contracts: AF-AFOSR-669-67 and FO 47-1-68-0001. The author wishes to acknowledge the many stimulating suggestions forwarded by the reviewers.

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☆ U.S. GOVERNMENT PRINTING OFFICE 1969 OL—354-455

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NAVAL RESEARCH
LOGISTICS
QUARTERLY

SEPTEMBER 1969
VOL. 16, NO. 3
NAVSO P-1278

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